A STUDY ON GENERAL SPLIT FEASIBILITY PROBLEM IN HILBERT SPACE VIA A RESEARCH OF M.ESLAMIAN ET. AL.

KANTIMA NORKAEW
CHIRATTIKARN CHANPROM
JUREEPORN PHINO

An Independent Study Submitted in Partial Fulfillment
of the Requirements for the Degree of Bachelor
of Science Program in Mathematics
December 2016
University of Phayao

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Advisor and Dean of School of Science have considered the independent study entitled "A study on general split feasibility problem in Hilbert space via a research of M.Eslamian et al." submitted in partial fulfillment of the requirements for the degree of Bachelor of Science Program in Mathematics is hereby approved.

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December 2016

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Kantima Nokeaw

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Jureeporn Phino

ชื่อเรื่อง	ปัญหาความเป็นไปได้แบบแยกส่วนในปริภูมิฮิวเบิร์ต โดยผ่านงานวิจัยของ M.Eslamian และคณะ
ผู้ศึกษาค้นคว้า	นางสาวกานติมา หน่อแก้ว นางสาวจิรัตน์ติกานต์ จันทร์พร่ำ นางสาวจุรีพร พิโน
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คำสำคัญ	จุดตรึง ปัญหาความเป็นไปได้แบบแยกส่วนทั่วไป ปริภูมิฮิวเบิร์ต การลู่เข้าแบบเข้ม

บทคัดย่อ

ในงานวิจัยนี้ เราได้ศึกษา เรื่อง ปัญหาความเป็นไปได้แบบแยกส่วนในปริภูมิฮิวเบิร์ต โดยเราได้ทำการศึกษาทฤษฎีบทการลู่เข้าผ่านงานวิจัยของ Eslamain และคณะ และ ขยายขั้นตอนการพิสูจน์ ให้มีความละเอียดและเข้าใจมากยิ่งขึ้น และยังได้ทำการยกตัวอย่าง ที่เกี่ยวข้องกับทฤษฎีที่ศึกษา เพื่อสนับสนุนว่าทฤษฎีนี้เป็นจริง

Title	A STUDY ON GENERAL SPLIT FEASIBILITY PROBLEM IN HILBERT SPACE VIA A RESEARCH OF M.ESLAMIAN ET. AL.
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ABSTRACT

In this paper, we studied subject of general split feasibility problem in Hilbert space together with the strong convergence theorem proposed by Mohammad Eslamian and Abdul Latif and expansion the proof line in each step to make it easier to understand. Furthermore, we give an example of algorithm which satisfies all the conditions in the main theorem and test its strong convergence.

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CHAPTER I

Introduction

In this paper, we studied subject of the general split feasibility problem in Hilbert space together with the strong convergence theorem proposed by Mohammad Eslamian and Abdul Latif [26] and expansion the proof line in each step to make it easier to understand. Furthermore, we give an example of the algorithm which satisfies all the conditions in the main theorem and tests its strong convergence.

Let H and K be infinite-dimensional real Hilbert spaces, and let $A : H \to K$ be a bounded linear operator. Let $\{C_i\}_{i=1}^{p}$ and $\{Q_i\}_{i=1}^{r}$ be the families of nonempty closed convex subsets of H and K, respectively.

Our mentioned problem is a general split feasibility problem (GSFP) which is to find a point x^*, which

$$x^* \in \bigcap_{i=1}^{\infty} C_i \quad \text{and} \quad Ax^* \in \bigcap_{i=1}^{\infty} Q_i. \tag{1.1}$$

We denote by Ω the solution set of GSFP.

The following show some problem which correspond to our study.

(a) The convex feasibility problem (CFP) is formulated as the problem of finding a point x^* with the property:

$$x^* \in \bigcap_{i=1}^{p} C_i.$$

(b) The split feasibility problem (SFP) is formulated as the problem of finding a point x^* with the property:

$$x^* \in C \quad \text{and} \quad Ax^* \in Q,$$

where C and Q are nonempty closed convex subsets of H and K, respectively.

(c) The multiple-set split feasibility problem (MSSFP) is formulated as the problem of finding a point x^* with the property:

$$x^* \in \bigcap_{i=1}^{p} C_i \quad \text{and} \quad Ax^* \in \bigcap_{i=1}^{r} Q_i.$$

We can see that (MSSFP) can be reduced to (SEP) if we take $p = r = 1$.

These problems can be applied in other disciplines such as the applications of CFP in image restoration, computer tomograph and radiation therapy treatment planning [1]. About soluing SFP, Censor and Elfving [2] recommended algorithm in the setting of finite-dimensional Hilbert spaces and pointed out that SFP can be used for soluing modelling inverse problems which arise from phase retrievals and in medical image reconstruction [3]. Recently, it is found that the SFP can also be applied to study the intensity-modulated radiation therapy; see, for example,[6,16] and the references therein. Since then, a lot of work has been done for finding a solution of SFP and MSSFP; see, for example,[2–25]. Very recently, Xu [8] considered the SFP in the setting of infinite-dimensional Hilbert spaces.

In [2], to solve the problem it depends on the existence of A^{-1}. In [8], Xu studied some algorithm and got both weak convergence and strong convergence theorems for solving the SFP by using Mann's algorithm and found that he can also get the solution of the minimum-norm. In [7], Wang and Xu proved their proposed cyclic algorithm as follow:

$$x_{n+1} = P_{C[n]}(x_n + yA^*(P_{Q[n]} - I)Ax_n),$$

where $[n] := n$ (mod p) , (mod function take values in $\{1, 2, ..., p\}$), and $\gamma \in (0, 2/\|A\|^2)$. They found that $\{x_n\}$ generated by their algorithm convergence weakly to the solution of MSSFP. Recently, they studied in case of the strongly convergence theorem for solving MSSFP in infinite dimensional Hilbert space, namely "General split feasibility problem (GSFP)" as follows which is to find x^* such that

$$x^* \in \bigcap_{i=1}^{\infty} C_i \quad and \quad Ax^* \in \bigcap_{i=1}^{\infty} Q_i$$

and denote its solution set as Ω.

Recently, Eslamian and Latif [26], proposed the viscosity iterative algorithm to solve GSFP in Hilbert space by improving all the results in the literature.

In this paper, we study the convergence theorem by Mohammad Eslamian and Abdul Latif whose steps are the solution to the GSFP:

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n, \quad n \geq 0,$$

then, the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega f(x^*)$.

CHAPTER II

Preliminaries and lemma

2.1 Preliminaries

Definition 2.1.1 ***[27] Linear operator***

A operator $T : X \to Y$ *from a vector space* X *to a vector space* Y *(with the same scalar field* K*) is a linear operator if:*

1. $T(x_1 + x_2) = T(x_1) + T(x_2),\ \forall x_1, x_2 \in X$,

2. $T(cx) = cT(x),\ \forall x \in X,\ c \in K$.

We call such transformations linear operators.

Definition 2.1.2 ***[32] Bounded operator***

A bounded operator $T : X \to Y$ *between two Banach spaces satisfies the inequality*

$$\|Tx\| \leq c\|x\|,$$

where c *is a constant independent of the choice of* $x \in X$.

Definition 2.1.3 ***[32] Adjoint operator***

The adjoint operator T^* *of an operator* T *in a Hilbert space* H *is an operator such that*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle,$$

for all x *and* y *in* H.

Definition 2.1.4 ***[28] Fixed point***

The point x *is a fixed point of the mapping* T *if* $T(x) = x$.

Definition 2.1.5 ***[28] Normed space***

Let X *be a vector space. A map* $T : X \to \mathbb{R},\ x \mapsto \|x\|$ *is called a norm on* X *if*

(i) $\|x\| \geq 0$ *for all* $x \in X$ *and* $\|x\| = 0$ *if and only if* $x = 0$;

(ii) $\|\alpha x\| = |\alpha|\|x\|$ *for all* $x \in X$ *and* $\alpha \in \mathbb{K}$;

(iii) $\|x+y\| \leq \|x\|+\|y\|$ *for all* $x, y \in X$ *(triangle inequality).*

We call $(X\ ,\ \|.\|)$ *or simply* X *a normed space.*

Definition 2.1.6 ***[30] Inner product space***

An inner product space is a complex linear space X *which for any pair of element* x *and* y *in* X *there corresponds a complex number, denoted by* $\langle x, y\rangle$*, and called the inner product of* x *and* y*, with the following properties :*

(i) $\langle x, x\rangle \geq 0,\ \langle x, x\rangle = 0 \leftrightarrow x = 0$*;*

(ii) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$*;*

(iii) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$*;*

(iv) $\langle x, y\rangle = \overline{\langle y, x\rangle}$*.*

Definition 2.1.7 ***[31] Hilbert space***

Let H *be an inner product space. Then* H *is called a* ***Hilbert space*** *if for each bounded sequence* $\{x_n\}$ *of* H*, there exists a weakly convergent subsequence of* $\{x_n\}$*.*

Definition 2.1.8 ***[31] Closed set***

Let H *be a Hilbert space. A subset* C *of* H *is called a* ***closed set*** *if*

$$\{x_n\} \subset C \text{ and } x_n \to x \text{ imply } x \in C.$$

Definition 2.1.9 ***[29] Convex set***

Let C *be a subset of a Hilbert space* H *and scalar* $t \in (0,1)$ *then* C *is said to be* ***convex*** *if*

$$tx + (1-t)y \in C \text{ for all } x, y \in C$$

Definition 2.1.10 ***[27] Nonexpansive mapping***

The mapping $T : C \to C$ *is said to be* ***nonexpansive mapping*** *if*

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.$$

Definition 2.1.11 ***[30] Contraction***

Let H *be a Hilbert space. A mapping* $f : C \to C$ *is called a* ***contraction*** *on* H *if there is a positive real number* $\alpha < 1$ *such that for all* $x, y \in C$

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|,\ \forall x, y \in H.$$

Definition 2.1.12 *[29] Strong convergence*

A sequence $\{x_n\}$ *in Hilbert space* H *is said to be* ***strongly convergence*** *(or convergence in the norm) if there is an* $x \in H$ *such that*

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

2.2 Lemma

Throughout this independent study, we denote by H a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $\{x_n\}$ be a sequence in H and $x \in H$. Weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$, and strong convergence by $x_n \to x$, respectively. Let C be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by P_Cx. This point satisfies

$$\|x - P_Cx\| \le \|x - y\|, \quad \forall y \in C. \tag{2.2}$$

The operator P_C is called the **metric projection** or the nearest point mapping of H onto C. The metric projection P_C is characterized by the fact that $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x)\rangle \leqslant 0, \quad \forall x \in H, y \in C. \tag{2.3}$$

It is well known that P_C is a nonexpansive mapping. It is also known that H satisfies Opial's condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.4}$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.2.13 *(see[21]). Let* H *be a Hilbert space. Then, for all* $x, y \in H$

$$\|x + y\|^2 \le \|x\|^2 + \langle y, x + y\rangle. \tag{2.5}$$

Lemma 2.2.14 *(see[22]). Let* H *be a Hilbert space, and let* $\{x_n\}$ *be a sequence in* H*. Then, for any given sequence* $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ *with* $\sum_{n=1}^{\infty} \lambda_n = 1$ *and for any positive integer* i, j *with* $i < j$*,*

$$\|\sum_{n=1}^{\infty} \lambda_n x_n\|^2 \le \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i\lambda_j\|x_i - x_j\|^2. \tag{2.6}$$

Lemma 2.2.15 *(see [23]). Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - y_n)a_n + y_n\delta_n + \beta_n, \quad n \geq 0, \tag{2.7}$$

where $\{\gamma_n\}, \{\beta_n\}$ *and* $\{\lambda_n\}$ *satisfy the following conditions:*

(i) $\gamma_n \subset [0, 1], \sum_{n=1}^{\infty} \gamma_n = \infty,$

(ii) lim sup $_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n\delta_n| < \infty,$

(iii) $\beta_n \geq 0$ *for all* $n \geq 0$ *with* $\sum_{n=1}^{\infty} \beta_n < \infty.$

Then, $\lim_{n\to\infty} a_n = 0.$

Lemma 2.2.16 *(see [24]). Let* $\{t_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $t_{n_j} < t_{n_i+1}$ *for all* $i \in \mathbb{N}$*. Then, there exists a nondecreasing sequence* $\{\tau(n)\} \subset \mathbb{N}$ *such that* $\tau(n) \to \infty$*, and the following properties are satisfied by all (sufficiently large) numbers* $n \in \mathbb{N}$*:*

$$t_{\tau(n)} \leq t_{\tau(n)+1}, \quad t_n \leq t_{\tau(n)+1}. \tag{2.8}$$

In fact

$$\tau(n) = \max\{k \leq n : t_k < t_{k+1}\}. \tag{2.9}$$

Lemma 2.2.17 *(**Demiclosedness principle** [25]). Let* C *be a nonempty closed and convex subset of a real Hilbert space* H*. Let* $T : C \to C$ *be a nonexpansive mapping such that* $Fix(T) \neq \emptyset$*. Then,* T *is demiclosed on* C*, that is, if* $y_n \rightharpoonup z \in C$*, and* $(y_n - Ty_n) \to y$ *then* $(I - T)z = y$*.*

CHAPTER III

Main result

In this chapter, we propose the study of general split feasibility problems in hilbert spaces, which is the method for solving GSFP. We expand all the proof lines in their theorem and give a numerical example which can explain how this algorithm can solve the mentioned problem.

Theorem 3.1.18 [26] *Let H and K be real Hilbert spaces, and Let $A : H \to K$ be a bounded linear operator. Let $\{C_i\}_{i=1}^{\infty}$ and $\{Q_i\}_{i=1}^{\infty}$ be the families of nonempty closed convex subset of H and K, respectively. Assume that GSFP(1.1) has a nonempty solution set Ω. Suppose that f is a self $k-contraction$ mapping of H, and let $\{x_n\}$ be a sequence generated by $x_0 \in H$ as*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n, \quad n \geq 0,$$

where $\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$. If the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\}$ and $\{\lambda_{n,i}\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$

(ii) for each $i \in \mathbb{N}, \liminf_n \alpha_n \gamma_{n,i} > 0$,

(iii) for each $i \in \mathbb{N}, \{\lambda_{n,i}\} \subset (0, 2/\|A\|^2)$ and

$0 < \liminf_{n\to\infty} \leq \limsup_{n\to\infty} \lambda_{n,i} < 2/\|A\|^2$,

then, the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega f(x^*)$.

Proof. First, we show that $\{x_n\}$ is bounded. In fact, let $z \in \Omega$. Since $\{\lambda_{n,i}\}$ $\subset (0, 2/\|A\|^2)$, the operators $P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)$ are nonexpansive, and hence we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n f(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - z\| \\
&= \|\alpha_n x_n + \alpha_n z - \alpha_n z + \beta_n f(x_n) + \beta_n z - \beta_n z \\
&\quad + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - z\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\alpha_n x_n - \alpha_n z\| + \|\beta_n f(x_n) - \beta_n z\| \\
&\quad + \|\alpha_n z + \beta_n z + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - z\| \\
&= \alpha_n \|x_n - z\| + \beta_n \|f(x_n) - z\| \\
&\quad + \|\sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - \sum_{i=1}^{\infty} \gamma_{n,i}(z)\| \\
&\leq \alpha_n \|x_n - z\| + \beta_n \|f(x_n) - z\| \\
&\quad + \sum_{i=1}^{\infty} \gamma_{n,i} \|P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - z\| \\
&= \alpha_n \|x_n - z\| + \beta_n \|f(x_n) - z\| + \sum_{i=1}^{\infty} \gamma_{n,i} \|x_n - z\| \\
&\leq (\alpha_n + \sum_{i=1}^{\infty} \gamma_{n,i}) \|x_n - z\| + \beta_n \|f(x_n) - z\| \\
&\leq (1 - \beta_n) \|x_n - z\| + \beta_n \|f(x_n) - z\| \\
&\leq (1 - \beta_n) \|x_n - z\| + \beta_n \|f(x_n) + f(z) - f(z) - z\| \\
&= (1 - \beta_n) \|x_n - z\| + \|\beta_n f(x_n) + \beta_n f(z) - \beta_n f(z) - \beta_n z\| \\
&\leq (1 - \beta_n) \|x_n - z\| + \|\beta_n f(x_n) - \beta_n f(z)\| + \|\beta_n f(z) - \beta_n z\| \\
&\leq (1 - \beta_n) \|x_n - z\| + \beta_n \|f(x_n) - f(z)\| + \beta_n \|f(z) - z\| \\
&\leq (1 - \beta_n) \|x_n - z\| + \beta_n k \|x_n - z\| + \beta_n \|f(z) - z\| \\
&\leq (1 - \beta_n + \beta_n k) \|x_n - z\| + \beta_n \|f(z) - z\| \\
&= (1 - (\beta_n - \beta_n k)) \|x_n - z\| + \beta_n \|f(z) - z\| \\
&= (1 - (1 - k)\beta_n) \|x_n - z\| + \beta_n \|f(z) - z\| \\
&\leq (1 - (1 - k)\beta_n) \|x_n - z\| + (1 - k) \frac{\beta_n}{(1 - k)} \|f(z) - z\| \\
&\leq \max\{\|x_n - z\|, \frac{\|f(z) - z\|}{1 - k}\} \\
&\quad \vdots \\
&\leq \max\{\|x_0 - z\|, \frac{\|f(z) - z\|}{1 - k}\} \qquad (3.10)
\end{aligned}
$$

which implies that $\{x_n\}$ is bounded and we also obtain that $\{f(x_n)\}$ is bounded. Next, we show that for each $i \in \mathbb{N}$,

$$
\lim_{n \to \infty} \|x_n - P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n\| = 0. \qquad (3.11)
$$

By using Lemma 2.2.14, for every $z \in \Omega$ and $i \in \mathbb{N}$, we have that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n x_n + \beta_n f(x_n) \\
&\quad + \sum_{j=1}^{\infty} \gamma_{n,j} P_{C_j}(I - \lambda_{n,j} A^*(I - P_{Q_j})A)x_n - z\|^2 \\
&\leq \alpha_n \|x_n - z\|^2 + \beta_n \|f(x_n) - z\|^2 \\
&\quad + \sum_{j=1}^{\infty} \gamma_{n,j} \|P_{C_j}(I - \lambda_{n,j} A^*(I - P_{Q_i})A)x_n - z\|^2 \\
&\quad - \alpha_n \gamma_{n,i} \|P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - x_n\| \\
&\leq \alpha_n \|x_n - z\|^2 + \beta_n \|f(x_n) - z\|^2 + \sum_{j=1}^{\infty} \gamma_{n,j} \|x_n - z\|^2 \\
&\quad - \alpha_n \gamma_{n,i} \|P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - x_n\|^2 \\
&\leq (1 - \beta_n)\|x_n - z\|^2 + \beta_n \|f(x_n) - z\|^2 \\
&\quad - \alpha_n \gamma_{n,i} \|P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - x_n\|^2. \qquad (3.12)
\end{aligned}$$

Hence, for each $i \in \mathbb{N}$ we have
$\alpha_n \gamma_{n,i} \|P_{C_i}(I - P_{Q_i})A)x_n - x_n\|^2$

$$\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \beta_n \|f(x_n) - z\|^2. \qquad (3.13)$$

Next, we show that there exists a unique $x^* \in \Omega$ such that $x^* = P_\Omega f(x^*)$. We observe that for each $n \geq 0, x^* \in \Omega$ solves the GSFP (1.1) if and only if x^* solves the fixed point equation

$$x^* = P_{c_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x^*, \quad i \in \mathbb{N}. \qquad (3.14)$$

that is, the solution sets of fixed point equation (20) and GSFP (1.1) are the same (see for details [8]). Note that if $\{\lambda_{n,i}\} \subset (0, 2/\|A\|^2)$, then the operators $P_{c_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)$ are nonexpansive.
$\|P_{c_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x - P_{c_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)y\|$

$$\begin{aligned}
&\leq \|(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x - (I - \lambda_{n,i} A^*(I - P_{Q_i})A)y\| \\
&\leq \|(x - \lambda_{n,i} A^*(I - P_{Q_i})Ax) - (y - \lambda_{n,i} A^*(I - P_{Q_i})Ay)\| \\
&\leq \|(x - \lambda_{n,i} A^* Ax + \lambda_{n,i} A^* P_{Q_i} Ax) - (y - \lambda_{n,i} A^* Ay + \lambda_{n,i} A^* P_{Q_i} Ay)\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \|x - \lambda_{n,i}x + \lambda_{n,i}A^*P_{Q_i}Ax - y + \lambda_{n,i}y - \lambda_{n,i}A^*P_{Q_i}Ay\| \\
&\leq \|(x-y) - \lambda_{n,i}(x-y)\| + \|\lambda_{n,i}A^*P_{Q_i}A(x-y)\| \\
&= (1-\lambda_{n,i})\|x-y\| + \lambda_{n,i}\|A^*P_{Q_i}A(x-y)\| \\
&\leq (1-\lambda_{n,i})\|x-y\| + \lambda_{n,i}\|P_{Q_i}A(x-y)\| \\
&\leq (1-\lambda_{n,i})\|x-y\| + \lambda_{n,i}\|A(x-y)\| \\
&\leq (1-\lambda_{n,i})\|x-y\| + \lambda_{n,i}\|x-y\| \\
&= (1-\lambda_{n,i}+\lambda_{n,i})\|x-y\| \\
&= \|x-y\|.
\end{aligned}
$$

So that

$$\|P_{c_i}(I - \lambda_{n,i}A^*(I - P_{Q_i})A)x - P_{c_i}(I - \lambda_{n,i}A^*(I - P_{Q_i})A)y\| \leq \|x-y\|.$$

Since the fixed point set of nonexpansive operators is closed and convex, the projection onto the solution set Ω is well defined whenever $\Omega \neq \emptyset$.

$$\|P_\Omega(f)(x) - P_\Omega(f)(y)\| \leq \|f(x) - f(y)\| \leq k\|x-y\|. \tag{3.15}$$

It is obvious that $P_\Omega(f)$ is a contraction of H into itself and actually, we can claim that P_Ω is nonexpansive. Hence, there exists a unique element x^* such that $x^* = P_\Omega f(x^*)$.

In order to prove that $x_n \to x^*$ as $n \to \infty$, we consider two possible cases.

Case 1. Assume that $\{\|x_n - x^*\|\}$ is a monotone sequence. In other words, for n_0 large enough, $\{\|x_n - x^*\|\}_{n \geq n_0}$ is either nondecreasing or nonincreasing. Since $\|x_n - x^*\|$ is bounded we have $\|x_n - x^*\|$ is convergent. Since $\lim_{n\to\infty}\beta_n = 0$ and $\{f(x_n)\}$ is bounded, from (3.13) we get that

$$\lim_{n\to\infty} \alpha_n\gamma_{n,i}\|P_{C_i}(I - \lambda_{n,i}A^*(I - P_{Q_i})A)x_n - x_n\|^2 = 0. \tag{3.16}$$

By assuming that $\liminf_n \lambda_n\gamma_{n,i} > 0$, we obtain

$$\lim_{n\to\infty} \|P_{C_i}(I - \lambda_{n,i}A^*(I - P_{Q_i})A)x_n - x_n\| = 0, \quad \forall i \in \mathbb{N}. \tag{3.17}$$

Now, we show that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \leq 0. \tag{3.18}$$

To show this inequality, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty}\langle f(x^*) - x^*, x_{n_k} - x^*\rangle = \limsup_{n\to\infty}\langle f(x*) - x^*, x_n - x^*\rangle. \tag{3.19}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to ω. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ and $\lambda_{n,i} \to \lambda_i \in (0, 2/\|A\|^2)$ for each $i \in \mathbb{N}$. From(3.17), we have

$$\begin{aligned}
&\|P_{C_i}(I - \lambda_i A^*(I - P_{Q_i})A)x_n - x_n\| \\
&= \|P_{C_i}(I - \lambda_i A^*(I - P_{Q_i})A)x_n - x_n \\
&\quad -P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n \\
&\quad +P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n\| \\
&\leq \|P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n \\
&\quad -P_{C_i}(I - \lambda_i A^*(I - P_{Q_i})A)x_n\| \\
&\quad +\|P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - x_n\| \\
&\leq \|(I - \lambda_i A^*(I - P_{Q_i})A)x_n - x_n \\
&\quad -(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n\| + \|P_{C_i} \\
&\quad (I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - x_n\| \\
&\leq |\lambda_i - \lambda_{n,i}| \|A^*(I - P_{Q_i})AX_n\| \\
&\quad +\|P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - x_n\| \qquad (3.20) \\
&\to 0 \ as \ n \to \infty.
\end{aligned}$$

Notice that for each $i \in \mathbb{N}, P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)$ is nonexpansive. Thus, from Lemma 2.2.17, we have $\omega \in \Omega$. Therefore, it follows that

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^* \rangle &= \lim_{k\to\infty} \langle f(x^*) - x^*, x_{n_k} - x^* \rangle = \langle f(x^*) - x^*, \omega - x^* \rangle \\
&\leq 0. \qquad (3.21)
\end{aligned}$$

Finally, we show that $x_n \to P_\Omega f(x^*)$. Applying Lemma 2.2.13, we have that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n x_n + \beta_n f(x) \\
&\quad + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - x^*\|^2 \\
&= \|\alpha_n x_n + \beta_n f(x) + \alpha_n x^* - \alpha_n x^* + \beta_n x^* - \beta_n x^* \\
&\quad + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - x^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \|(\alpha_n x_n - \alpha_n x^*) + (\beta_n f(x) - \beta_n x^*) + (\alpha_n x^* + \beta_n x^* - x^*) \\
&\quad + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - x^*\|^2 \\
&\leq \|\alpha(x_n - x^*) + \beta_n(f(x) - x^*) \\
&\quad +(\sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - \sum \gamma_{n,i} x^*)\|^2 \\
&\leq \|\alpha_n(x_n - x^*) + \beta_n(f(x) - x^*) \\
&\quad + \sum \gamma_{n,i}(P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - x^*)\|^2 \\
&\leq \|\alpha_n(x_n - x^*) + \sum \gamma_{n,i}(P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - x^*) \\
&\quad +\beta_n(f(x) - x^*)\|^2 \\
&\leq \|\alpha_n(x_n - x^*) + \sum \gamma_{n,i}(P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - x^*)\|^2 \\
&\quad +2\langle \beta_n(f(x) - x^*), x_{n+1} - x^* \rangle \\
&\leq \|\alpha_n(x_n - x^*) + \sum \gamma_{n,i}(P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n - x^*)\|^2 \\
&\quad +2\beta_n\langle f(x) - x^*, x_{n+1} - x^* \rangle \\
&\leq \|\alpha_n(x_n - x^*) + \sum \gamma_{n,i}(x_n - x^*)\|^2 \\
&\quad +2\beta_n\langle f(x_n) - x^*, x_{n+1} - x^* \rangle \\
&\leq \|(1 - \beta_n)x_n - x^*\|^2 + 2\beta_n\langle f(x_n) - x^*, x_{n+1} - x^* \rangle \\
&\leq (1 - \beta_n)^2\|x_n - x^*\|^2 + 2\beta\langle f(x_n) - x^*, x_{n+1} - x^* \rangle \\
&\leq (1 - \beta_n)^2\|x_n - x^*\|^2 + 2\beta_n\langle f(x_n) - f(x^*), x_{n+1} - x^* \rangle \\
&\quad +2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\leq (1 - \beta_n)^2\|x_n - x^*\|^2 + 2\beta_n\|f(x_n) - f(x^*)\|\|x_{n+1} - x^*\| \\
&\quad +2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\
&\leq (1 - \beta_n)^2\|x_n - x^*\|^2 + 2\beta_n k\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad +2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^* \rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq\ & (1 - \beta_n)^2\|x_n - x^*\|^2 + 2\beta_n k\|x_n - x^*\|\|x_{n+1} - x^*\| \\
& +2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^* \rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\beta_n)^2\|x_n - x^*\|^2 + \beta_n k\|x_n - x^*\|^2 + \beta_n k\|x_{n+1} - x^*\|^2 \\
&\quad +2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle. \\
&\leq (1-\beta_n)^2\|x_n - x^*\|^2 + \beta_n k\{\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\} \\
&\quad +2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 - \beta_n k\|x_{n+1} - x^*\|^2 &\leq (1-\beta_n)^2\|x_n - x^*\|^2 + \beta_n k\|x_n - x^*\|^2 \\
&\quad +2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle.
\end{aligned}
$$

That is

$$
\begin{aligned}
(1-\beta_n k)\|x_{n+1} - x^*\|^2 &\leq (1-\beta_n)^2\|x_n - x^*\|^2 + \beta_n k\|x_n - x^*\|^2 \\
&\quad +2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \frac{(1-\beta_n)^2 + \beta_n k\|x_n - x^*\|^2}{1-\beta_n k} \\
&\quad + \frac{2\beta_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle}{1-\beta_n k} \\
&\leq \frac{(1-\beta_n)^2 + \beta_n k}{1-\beta_n k}\|x_n - x^*\|^2 \\
&\quad + \frac{2\beta_n}{1-\beta_n k}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= \frac{(1-2\beta_n + \beta_n^2) + \beta_n k}{1-\beta_n k}\|x_n - x^*\|^2 \\
&\quad + \frac{2\beta_n}{1-\beta_n k}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= \frac{1-2\beta_n + \beta_n k}{1-\beta_n k}\|x_n - x^*\|^2 \\
&\quad + \frac{\beta_n^2}{1-\beta_n k}\|x_n - x^*\|^2 \\
&\quad + \frac{2\beta_n}{1-\beta_n k}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle
\end{aligned}
$$

$$\begin{aligned}
&\leq (1-\frac{2(1-k)\beta_n}{1-\beta_n k})\|x_n-x^*\|^2 \\
&\quad +\frac{2(1-k)\beta_n}{1-\beta_n k}\{\frac{\beta_n M}{2(1-k)} \\
&\quad +\frac{1}{1-k}\langle f(x^*)-x^*,x_{n+1}-x^*\rangle\} \\
&\leq (1-\eta_n)\|x_n-x^*\|^2+\eta_n\delta_n
\end{aligned}$$

where $\delta_n = \frac{\beta n M}{2(1-k)} + \frac{1}{1-k}\langle f(x^*)-x^*,x_{n+1}-x^*\rangle$ and $M=\sup\{\|x_n-x^*\|^2 : n\geq 0\}$ and $\eta_n = 2(1-k)\beta_n/(1-\beta_n k)$.

It is easy to see that $\eta_n \to 0$, $\sum_{n=1}^{\infty}\eta_n=\infty$ and $\limsup_{n\leftarrow\infty}\delta_n\leq 0$. Hence, by Lemma 2.2.15, the sequence $\{x_n\}$ converges strongly to $x^*=P_\Omega f(x^*)$.

Case 2. Assume that $\{x_n-x^*\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n\geq n_0$ (for some n_0 large enough) by

$$\tau(n)=\max\{k\in\mathbb{N}; k\leq n : \|x_k-x^*\|<\|x_{k+1}-x^*\|\}.$$

Clearly, $\tau(n)$ is a nondecreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and for all $n\geq n_0$,

$$\|x_{\tau(n)}-x^*\|<\|x_{\tau(n)+1}-x^*\|.$$

From (3.13), we obtain that

$$\lim_{n\to\infty}\|P_{C_i}(I-\lambda_{\tau(n),i}A^*(I-P_{Qi})A)x_{\tau(n)}-x_{\tau(n)}\|=0.$$

Following an argument similar to that in Case 1, we have

$$\limsup_{n\to\infty}\langle f(x^*)-x^*,x_{\tau(n)+1}-x^*\rangle\leq 0.$$

And by similar argument, we have

$$\begin{aligned}
\|x_{\tau(n)+1}-x^*\|^2 &= \|\alpha_{\tau(n)}x_{\tau(n)+1}+\beta_{\tau(n)}f(x_{\tau(n)}) \\
&\quad +\sum_{i=1}^{\infty}\gamma_{\tau(n),i}P_{C_i}(I-\lambda_{\tau(n),i}A^*(I-P_{Q_i})A)x_{\tau(n)}-x^*\|^2 \\
&= \|\alpha_{\tau(n)}x_{\tau(n)}+\beta_{\tau(n)}f(x_{\tau(n)})+\alpha_{\tau(n)}x^*-\alpha_{\tau(n)}x^*+\beta_{\tau(n)}x^* \\
&\quad -\beta_{\tau(n)}x^*+\sum_{i=1}^{\infty}\gamma_{\tau(n),i}P_{C_i}(I-\lambda_{\tau(n),i}A^*(I-P_{Q_i})A)x_{\tau(n)}-x^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \|(\alpha_{\tau(n)}x_{\tau(n)} - \alpha_{\tau(n)}x^*) + (\beta_{\tau(n)}f(x_{\tau(n)}) - \beta_{\tau(n)}x^*) \\
&\quad +(\alpha_{\tau(n)}x^* + \beta_{\tau(n)}x^* - x^*) \\
&\quad +\sum_{i=1}^{\infty}\gamma_{\tau(n),i}P_{C_i}(I - \lambda_{\tau(n),i}A^*(I - P_{Q_i})A)x_{\tau(n)} - x^*\|^2 \\
&\leq \|\alpha(x_{\tau(n)} - x^*) + \beta_{\tau(n)}(f(x_{\tau(n)}) - x^*) \\
&\quad +(\sum_{i=1}^{\infty}\gamma_{\tau(n),i}P_{C_i}(I - \lambda_{\tau(n),i}A^*(I - P_{Q_i})A)x_{\tau(n)} \\
&\quad -\sum_{i=1}^{\infty}\gamma_{\tau(n),i}x^*)\|^2 \\
&\leq \|\alpha_{\tau(n)}(x_{\tau(n)} - x^*) + \beta_{\tau(n)}(f(x_{\tau(n)}) - x^*) \\
&\quad +\sum_{i=1}^{\infty}\gamma_{\tau(n),i}(P_{C_i}(I - \lambda_{\tau(n),i}A^*(I - P_{Q_i})A)x_{\tau(n)} - x^*)\|^2 \\
&\leq \|\alpha_{\tau(n)}(x_{\tau(n)} - x^*) \\
&\quad +\sum_{i=1}^{\infty}\gamma_{\tau(n),i}(P_{C_i}(I - \lambda_{\tau(n),i}A^*(I - P_{Q_i})A)x_{\tau(n)} - x^*) \\
&\quad +\beta_{\tau(n)}(f(x_{\tau(n)}) - x^*)\|^2 \\
&\leq \|\alpha_{\tau(n)}(x_{\tau(n)} - x^*) \\
&\quad +\sum_{i=1}^{\infty}\gamma_{\tau(n),i}(P_{C_i}(I - \lambda_{\tau(n),i}A^*(I - P_{Q_i})A)x_{\tau(n)} - x^*)\|^2 \\
&\quad +2\langle\beta_{\tau(n)}(f(x) - x^*), x_{\tau(n)+1} - x^*\rangle \\
&\leq \|\alpha_{\tau(n)}(x_{\tau(n)} - x^*) \\
&\quad +\sum_{i=1}^{\infty}\gamma_{\tau(n),i}(P_{C_i}(I - \lambda_{\tau(n),i}A^*(I - P_{Q_i})A)x_{\tau(n)} - x^*)\|^2 \\
&\quad +2\beta_{\tau(n)}\langle f(x_{\tau(n)}) - x^*, x_{\tau(n)+1} - x^*\rangle \\
&\leq \|\alpha_{\tau(n)}(x_{\tau(n)} - x^*) + \sum_{i=1}^{\infty}\gamma_{\tau(n),i}(x_{\tau(n)} - x^*)\|^2 \\
&\quad +2\beta_{\tau(n)}\langle f(x_{\tau(n)}) - x^*, x_{\tau(n)+1} - x^*\rangle \\
&\leq \|(1 - \beta_{\tau(n)})x_{\tau(n)} - x^*\|^2 \\
&\quad +2\beta_{\tau(n)}\langle f(x_{\tau(n)}) - x^*, x_{\tau(n)+1} - x^*\rangle \\
&\leq (1 - \beta_n)^2\|x_{\tau(n)} - x^*\|^2 \\
&\quad +2\beta_{\tau(n)}\langle f(x_{\tau(n)}) - x^*, x_{\tau(n)+1} - x^*\rangle \\
&= (1 - \beta_{\tau(n)})^2\|x_{\tau(n)} - x^*\|^2 \\
&\quad +2\beta_{\tau(n)}\langle f(x_{\tau(n)}) - x^* + f(x^*) - f(x^*), x_{\tau(n)+1} - x^*\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\beta_{\tau(n)})^2\|x_{\tau(n)}-x^*\|^2 \\
&\quad +2\beta_{\tau(n)}\langle f(x_{\tau(n)})-f(x^*), x_{\tau(n)+1}-x^*\rangle \\
&\quad +2\beta_{\tau(n)}\langle f(x^*)-x^*, x_{\tau(n)+1}-x^*\rangle \\
&\leq (1-\beta_{\tau(n)})^2\|x_{\tau(n)}-x^*\|^2 \\
&\quad +2\beta_{\tau(n)}\|f(x_{\tau(n)})-f(x^*)\|\|x_{\tau(n)+1}-x^*\| \\
&\quad +2\beta_{\tau(n)}\langle f(x^*)-x^*, x_{\tau(n)+1}-x^*\rangle.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|x_{\tau(n)+1}-x^*\|^2 &\leq (1-\beta_{\tau(n)})^2\|x_{\tau(n)}-x^*\|^2 \\
&\quad +2\beta_{\tau(n)}k\|x_{\tau(n)}-x^*\|\|x_{\tau(n)+1}-x^*\| \\
&\quad +2\beta_{\tau(n)}\langle f(x^*)-x^*, x_{\tau(n)+1}-x^*\rangle \\
&\leq (1-\beta_{\tau(n)})^2\|x_{\tau(n)}-x^*\|^2 \\
&\quad +\beta_{\tau(n)}k\{\|x_{\tau(n)}-x^*\|^2+\|x_{\tau(n)+1}-x^*\|^2\} \\
&\quad +2\beta_{\tau(n)}\langle f(x^*)-x^*, x_{\tau(n)+1}-x^*\rangle \\
&\leq (1-\beta_{\tau(n)})^2\|x_{\tau(n)}-x^*\|^2+\beta_{\tau(n)}k\|x_{\tau(n)}-x^*\|^2 \\
&\quad +\beta_{\tau(n)}k\|x_{\tau(n)+1}-x^*\|^2 \\
&\quad +2\beta_{\tau(n)}\langle f(x^*)-x^*, x_{\tau(n)+1}-x^*\rangle.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\|x_{\tau(n)+1}-x^*\|^2-\beta_{\tau(n)}k\|x_{\tau(n)+1}-x^*\|^2 &\leq (1-\beta_{\tau(n)})^2\|x_{\tau(n)}-x^*\|^2 \\
&\quad +\beta_{\tau(n)}k\|x_{\tau(n)}-x^*\|^2 \\
&\quad +2\beta_{\tau(n)}\langle f(x^*)-x^*, x_{\tau(n)+1}-x^*\rangle.
\end{aligned}
$$

$$
\begin{aligned}
(1-\beta_{\tau(n)}k)\|x_{\tau(n)+1}-x^*\|^2 &\leq (1-\beta_{\tau(n)})^2\|x_{\tau(n)}-x^*\|^2+\beta_{\tau(n)}k\|x_{\tau(n)}-x^*\|^2 \\
&\quad +2\beta_{\tau(n)}\langle f(x^*)-x^*, x_{\tau(n)+1}-x^*\rangle \\
&\leq \frac{(1-\beta_{\tau(n)})^2+\beta_{\tau(n)}k\|x_{\tau(n)}}{1-\beta_{\tau(n)}k} \\
&\quad -\frac{x^*\|^2+2\beta_{\tau(n)}\langle f(x^*)-x^*, x_{\tau(n)+1}-x^*\rangle}{1-\beta_{\tau(n)}k}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{(1-\beta_{\tau(n)})^2+\beta_{\tau(n)}k}{1-\beta_n k}\|x_n-x^*\|^2\\
&\quad+\frac{2\beta_{\tau(n)}}{1-\beta_{\tau(n)}k}\langle f(x^*)-x^*, x_{\tau(n)+1}-x^*\rangle\\
&= \frac{(1-2\beta_{\tau(n)}+\beta^2_{\tau(n)})+\beta_{\tau(n)}k}{1-\beta_{\tau(n)}k}\|x_{\tau(n)}-x^*\|^2\\
&\quad+\frac{2\beta_{\tau(n)}}{1-\beta_{\tau(n)}k}\langle f(x^*)-x^*, x_{\tau(n)+1}-x^*\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1-2\beta_{\tau(n)}+\beta_{\tau(n)}k}{1-\beta_{\tau(n)}k}\|x_{\tau(n)}-x^*\|^2\\
&\quad+\frac{\beta^2_{\tau(n)}}{1-\beta_{\tau(n)}k}\|x_{\tau(n)}-x^*\|^2\\
&\quad+\frac{2\beta_{\tau(n)}}{1-\beta_{\tau(n)}k}\langle f(x^*)-x^*, x_{\tau(n)+1}-x^*\rangle\\
&\le (1-\frac{2(1-k)\beta_{\tau(n)}}{1-\beta_{\tau(n)}k})\|x_{\tau(n)}-x^*\|^2\\
&\quad+\frac{2(1-k)\beta_{\tau(n)}}{1-\beta_{\tau(n)}k}\{\frac{\beta_{\tau(n)}M}{2(1-k)}\\
&\quad+\frac{1}{1-k}\langle f(x^*)-x^*, x_{\tau(n)+1}-x^*\rangle\}\\
&\le (1-\eta_{\tau(n)})\|x_{\tau(n)}-x^*\|^2+\eta_{\tau(n)}\delta_{\tau(n)}.
\end{aligned}
$$

where $\eta_{\tau(n)} \to 0$, $\sum_{n=1}^{\infty}\eta_{\tau(n)} = \infty$ and $\limsup_{n\to\infty}\delta_{\tau(n)} \le 0$. Hence, by Lemma 2.2.15, we obtain $\lim_{n\to\infty}\|x_{\tau(n)}-x^*\| = 0$ and $\lim_{n\to\infty}\|x_{\tau(n)+1}-x^*\| = 0$.

Now, from Lemma 2.2.16, we have

$$
\begin{aligned}
0 &\le \|x_n-x^*\|\\
&\le \max\{\|x_{\tau(n)}-x^*\|, \|x_n-x^*\|\}\\
&\le \|x_{\tau(n)+1}-x^*\|,
\end{aligned}
$$

therefore, $\{x_n\}$ converges strongly to $x^* = P_\Omega f(x^*)$.

CHAPTER IV

Numerical example

In this chapter, we propose some numerical example which support the main theorem.

4.1 Numerical Example

Example 4.1.1 Let $H \equiv K \equiv \mathbb{R}$ and $C \in [0,1]$ and the other conditions as follow : $\alpha_n = \frac{1}{2} - (\frac{2n+1}{3n^2+2})$, $\beta_n = \frac{2n+1}{3n^2+2}$, $\sum_{i=1}^{\infty} \gamma_{n,i} = \frac{1}{2}$, $\lambda_{n,i} = \frac{n}{1+5n}$, $x_0 = 10$,
$f(x_n) = \frac{2}{3}(x_n)$, $Ax = \frac{x}{2}$, $A^*x = \frac{x}{2}$, $C_i = [0,1]$, $Q_i = [0,2]$.

First, we will check that all parameter satisfy all the condition in Main Theorem.
Setting

1. $\beta_n = \frac{2n+1}{3n^2+2}$,

since β_n has to satisfy the condition $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$

Consider the following,

$$\begin{aligned}
\lim_{n\to\infty} \beta_n &= \lim_{n\to\infty} \frac{2n+1}{3n^2+2} \\
&= \lim_{n\to\infty} \frac{\frac{2n}{n^2} + \frac{1}{n^2}}{\frac{3n^2}{n^2} + \frac{2}{n^2}} \\
&= 0.
\end{aligned}$$

Next, we show that $\sum_{n=1}^{\infty} \beta_n$ is divergent.

Let $\beta_n = \frac{2n+1}{3n^2+2}$ and $b_n = \frac{1}{n}$, consider

$$\begin{aligned}
\lim_{n\to\infty} \frac{\beta_n}{b_n} &= \lim_{n\to\infty} \frac{2n+1}{3n^2+2} \cdot \frac{n}{1} \\
&= \lim_{n\to\infty} \frac{2n^2+n}{3n^2+2} \\
&= \frac{2}{3}.
\end{aligned}$$

Therefore $\lim_{n\to\infty} \frac{\beta_n}{b_n} = \frac{2}{3}$.

Such that $\sum_{n=1}^{\infty} \beta_n = \infty$

2. $\sum_{i=1}^{\infty} \gamma_{n,i} = \frac{1}{2}$

Let $\gamma_{n,i} = \frac{1}{(2i-1)(2i+1)}$.

Check

$$S_n = \sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)}$$
$$= \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2n-1)(2n+1)}.$$

By investigation $\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}(\frac{1}{2n-1} - \frac{1}{2n+1})$, hence

$$S_n = \frac{1}{2}(1-\frac{1}{3}) + \frac{1}{2}(\frac{1}{3}-\frac{1}{5}) + ... + \frac{1}{2}(\frac{1}{2n-1} - \frac{1}{2n+1})$$
$$= \frac{1}{2}[(1-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{5}) + ... + \frac{1}{2}(\frac{1}{2n-1} - \frac{1}{2n+1})]$$
$$= \frac{1}{2}[1 + (-\frac{1}{3}+\frac{1}{3}) + (-\frac{1}{5}+\frac{1}{5}) + ... + (-\frac{1}{2n-1} + \frac{1}{2n-1}) - \frac{1}{2n+1}]$$
$$= \frac{1}{2}(1 - \frac{1}{2n+1}).$$

Therefore $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}(1-\frac{1}{2n+1}) = \frac{1}{2}(1-0) = \frac{1}{2}$.

Consequently $\sum_{i=1}^{\infty} \frac{1}{(2i-1)(2i+1)} = \frac{1}{2}$.

3. $\lambda_{n,i} = \frac{n}{1+5n}$

Since $\lambda_{n,i}$ has to satisfy the condition $\lambda_{n,i} \subset (0, 2/\|A\|^2)$ and $0 < \liminf_{n\to\infty} \lambda_{n,i}$ $\leq \limsup_{n\to\infty} \lambda_{n,i} < 2/\|A\|^2$. Consider the following,

$$\begin{aligned} \lim_{n\to\infty} \lambda_{n,i} &= \lim_{n\to\infty} \frac{n}{1+5n} \\ &= \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{1}{n}+\frac{5n}{n}} \\ &= \frac{1}{5}. \end{aligned}$$

Such that $\lambda_{n,i} = \frac{n}{1+5n}$.

4. $f(x_n) = \frac{2}{3}(x_n)$

Let $f(x) = \frac{2}{3}x$, next we have to show that

$\|f(x) - f(y)\| \leq k\|x-y\|; k \in (0,1)$.

Let $x, y \in H$

$$\|f(x) - f(y)\| = \|\tfrac{2}{3}x - \tfrac{2}{3}y\|$$
$$= \tfrac{2}{3}\|x-y\|.$$

Hence, f is a contraction.

5. Ax = $\frac{x}{2}$ (Bounded linear operator)

We show that 5.1. A is bounded operator

5.2. A is linear operator.

5.1 Let $Ax = \frac{x}{2}$

$$\begin{aligned} \|Ax\| &\leq c\|x\| \\ \|Ax\| &= \|\frac{x}{2}\| \\ &= \frac{1}{2}\|x\|. \end{aligned}$$

5.2 Next, we show that A is a linear operator.

From $Ax = \frac{x}{2}$, we have

5.2.1 $A(x+y) = \frac{x+y}{2}$

$= \frac{x}{2} + \frac{y}{2}$

$= Ax + Ay.$

5.2.2 $A(cx) = \frac{cx}{2}$

$= c(\frac{x}{2})$

$= cAx.$

So, $Ax = \frac{x}{2}$ is a Bounded linear operator.

6. $A^*x = \frac{x}{2}$ (Adjoint operator)

Consider $\langle Ax, y\rangle$ for all $x, y \in H$ as follows:

$\langle Ax, y\rangle$ = $\langle \frac{x}{2}, y\rangle$

= $\frac{1}{2}xy$

A^* be an adjoint operator of A, therefore

$$\langle Ax, y\rangle = \langle x, A^*y\rangle.$$

It means that $\langle x, A^*y\rangle = \frac{1}{2}xy$, then

$$A^*y = \frac{1}{2}y.$$

So, $A^*x = Ax = \frac{x}{2}$.

7. $C_i = [0, 1]$

where

$$P_{C_i}x = \begin{cases} 1, & x > 1, \\ 0, & x < 0, \\ x, & x \in [0, 1]. \end{cases}$$

8. $Q_i = [0, 2]$

where

$$P_{Q_i}x = \begin{cases} 2, & x > 2, \\ 0, & x < 0, \\ x, & x \in [0, 2]. \end{cases}$$

Next, we will construct the iteration process by follow the algorithm.

Hence,

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \beta_n f(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} P_{c_i}(x_n - \lambda_{n,i}(x_n - A^* P_{Q_i} A x_n)) \\ &= (\frac{1}{2} - (\frac{2n+1}{3n^2+2}))x_n + (\frac{2n+1}{3n^2+2})(\frac{2}{3}(x_n)) \\ &\quad + \frac{1}{2}(P_{C_i}(x_n - (\frac{n}{1+5n})(x_n - \frac{P_{Q_i}\frac{x_n}{2}}{2}))) \end{aligned}$$

After we run this algorithm by using Microsoft Excel with the setting $x_0 = 10$, we can get the value of x_n as following :

n	x_n	n	x_n
0	10	$\vdots$	$\vdots$
1	3.833333333	114	0.000175328
2	2.068181818	115	0.000162087
3	1.44338118	116	0.000149847
4	1.181113609	117	0.000138531
5	1.066287347	118	0.000128071
6	0.972168187	119	0.000118401
7	0.888685112	120	0.000109462
8	0.81385485	121	0.000101198

Table 1. The value of x_n generated by Example 4.1.1.

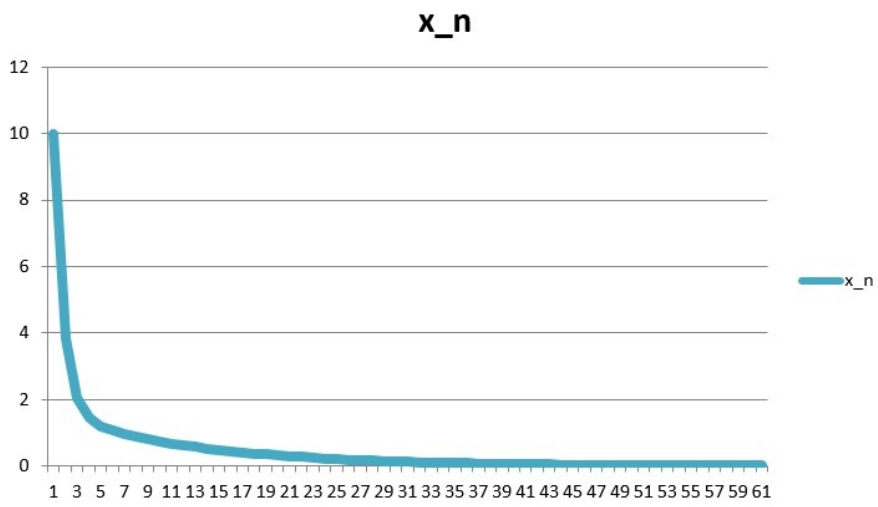

Figure 1. The convergence of $\{x_n\}$ by Example 4.1.1.

CHAPTER V

Conclusions

In this Chapter, we propose the conclusion of our study which consists of the main theorem, and numerical examples as shown in the followings.

Theorem 5.1.1. [26] Let H and K be real Hilbert spaces, and Let $A : H \rightarrow K$ be a bounded linear operator. Let $\{C_i\}_{i=1}^{\infty}$ and $\{Q_i\}_{i=1}^{\infty}$ be the families of nonemply closed convex subset of H and K, respectively. Assume that GSFP (1.1) has a nonempty solution set Ω. Suppose that f is a self $k-contraction$ mapping of H, and let $\{x_n\}$ be a sequence generated by $x_0 \in H$ as

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} P_{C_i}(I - \lambda_{n,i} A^*(I - P_{Q_i})A)x_n, \quad n \geq 0,$$

where $\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$. If the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\}$ and $\{\lambda_{n,i}\}$ satisfy the following conditions:

(i) $\lim_{n\rightarrow\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$

(ii) for each i $\in \mathbb{N}, \liminf_n \alpha_n \gamma_{n,i} > 0$,

(iii) for each i $\in \mathbb{N}, \{\lambda_{n,i}\} \subset (0, 2/\|A\|^2)$ and

$0 < \liminf_{n\rightarrow\infty} \leq \limsup_{n\rightarrow\infty} \lambda_{n,i} < 2/\|A\|^2$,

then, the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_{\Omega} f(x^*)$.

Next, we would like to show step proof of the theorem 5.1.1.

1. Find $\{x_n\}$ is bounded.
2. Find $\lim_{n\rightarrow\infty} \|x_n - P_{c_i}(I - \lambda_{n,i} a^*(I - P_{Q_i})A)x_n\| = 0$.
3. Show that there exists a unique $x^* \in \Omega$ such that $x^* = P_{\Omega} f(x^*)$.
4. Prove that $x_n \rightarrow x^*$ as $x \rightarrow \infty$.

   - Case 1 Assume that $\{\|x_n - x^*\|\}$ is a monotone sequence.
   - Case 2 Assume that $\{\|x_n - x^*\|\}$ is not a monotone sequence.

Example 5.1.1 Let $H \in \mathbb{R}$ and $C \in [0, 1]$ and the other conditions as follow :

$\alpha_n = \frac{1}{2} - (\frac{2n+1}{3n^2+2}), \quad \beta_n = \frac{2n+1}{3n^2+2}, \quad \sum_{i=1}^{\infty} \gamma_{n,i} = \frac{1}{2}, \quad \lambda_{n,i} = \frac{n}{1+5n}, \quad x_0 = 10,$

$f(x_n) = \frac{2}{3}(x_n), \quad Ax = \frac{x}{2}, \quad A^*x = \frac{x}{2}, \quad C_i = [0, 1], \quad Q_i = [0, 2].$

After we run this algorithm by using Microsoft Excel with the setting $x_0 = 10$, we can get the value of x_n as following :

n	x_n	n	x_n
0	10	$\vdots$	$\vdots$
1	3.833333333	114	0.000175328
2	2.068181818	115	0.000162087
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4	1.181113609	117	0.000138531
5	1.066287347	118	0.000128071
6	0.972168187	119	0.000118401
7	0.888685112	120	0.000109462
8	0.81385485	121	0.000101198

Table 2. The value of x_n generated by Example 5.1.1.

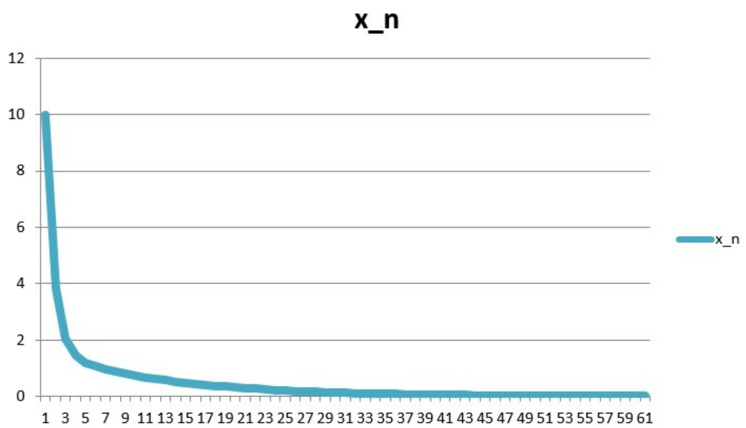

Figure 2. The convergence of $\{x_n\}$ by Example 5.1.1.

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