A SELF-ADAPTIVE METHOD FOR SOLVING THE SPLIT FEASIBILITY PROBLEM AND THE FIXED POINT PROBLEM OF BREGMAN STRONGLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ชื่อเรื่อง	วิธีการเซลฟ์-อแดปทีฟ สำหรับแก้ปัญหาความเป็นไปได้แบบแยกส่วนและปัญหาจุดตรึงของการส่งแบบไม่ขยายอย่างเข้มเบรกแมนในปริภูมิบานาค
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ในงานวิจัยนี้ เราเสนอวิธีการเซลฟ์-อแดปทีฟแบบใหม่ในการหาคำตอบร่วมของปัญหาความเป็นไปได้แบบแยกส่วนและปัญหาจุดตรึงของการส่งแบบไม่ขยายอย่างเข้มเบรกแมน เราได้พิสูจน์ทฤษฎีบทการลู่เข้าแบบเข้มภายใต้เงื่อนไขบางอย่างที่เหมาะสม นอกจากนี้เรายังได้ยกตัวอย่างผลลัพธ์เชิงตัวเลขเพื่อแสดงถึงประสิทธิภาพและการนำไปใช้งานของวิธีการดังกล่าว

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ABSTRACT

In this work, we suggest a new self-adaptive method for finding a common solution of the split feasibility problem and the fixed point problem of Bregman strongly nonexpansive mappings. We prove its strong convergence theorem under some mild conditions. We also give some numerical examples to show the efficiency and implementation of our method.

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CHAPTER I

Introduction

Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively; Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A which is defined by

$$\langle A^* \bar{y}, x \rangle := \langle \bar{y}, Ax \rangle, \ \forall x \in E_1, \bar{y} \in E_2^*.$$

The $split\ feasibility\ problem$ (SFP) is to find a point

$$x \in C \quad \text{such} \quad \text{that } Ax \in Q. \tag{1.1.1}$$

We denote by $\Omega = C \cap A^{-1}(Q) = \{y \in C : Ay \in Q\}$ the solution set of SFP. Then we have that Ω is a closed and convex subset of E_1.

The SFP in finite-dimensional Hilbert spaces was introduced by Censor and Elfving [8] for modelling inverse problems which arise from phase retrievals, medical image reconstruction and recently in modelling of intensity modulated radiation therapy. The SFP attracts the attention of many authors due to its application in signal processing. Various algorithms and some interesting results have been invented to solve it (see, for example, [1, 3, 4, 6, 14, 18, 19, 20, 30]).

For solving SFP, in p-uniformly convex and uniformly smooth real Banach spaces, Schöpfer et al [24] proposed the following algorithm: For $x_1 \in E_1$ and

$$x_{n+1} = \Pi_C J_{E_1}^* [J_{E_1}(x_n) - t_n A^* J_{E_2}(Ax_n - P_Q(Ax_n))], \quad n \geq 1, \tag{1.1.2}$$

where Π_C denotes the Bregman projection and J the duality mapping. Clearly, the above algorithm covers the CQ-algorithm which was introduced by Byrne [7], which is defined by

$$x_{n+1} = P_C(x_n - \mu_n A^*(I - P_Q)Ax_n), \quad n \geq 1, \tag{1.1.3}$$

where $\mu_n \in (0, \frac{2}{\|A\|^2})$ and P_C, P_Q are the metric projections on C and Q, respectively, which is found to be a gradient-projection method in convex minimization as a special case. It was proved that $\{x_n\}$ defined by (1.1.3) converges weakly to a solution of SFP.

We observe that the operator norm $\|A\|$ may not be calculated easily in general. To overcome this difficulty, López et al. [14] suggested the following self-adaptive method, which permits step-size μ_n being selected self-adaptively in such a way:

$$\mu_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \quad n \geq 1, \tag{1.1.4}$$

where $\rho_n \in (0, 4)$, $f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ and $\nabla f(x_n) = A^*(I - P_Q)Ax_n$ for all $n \geq 1$. It was proved that the sequence $\{x_n\}$ defined by (1.1.4) converges weakly to a solution of SFP.

Also, employing the idea of Halpern's iteration, López et al. [14] proposed the following iteration method:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)P_C(x_n - \mu_n \nabla f(x_n)), \quad n \geq 1, \tag{1.1.5}$$

where $\{\alpha_n\} \subset [0, 1]$, $u \in C$ and the step-size μ_n is chosen as above. It was proved that $\{x_n\}$ defined by (1.1.5) converges strongly to a solution of SFP provided $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty} \alpha_n = \infty$. After that, there have been many modifications of the CQ algorithm and the self-adaptive method established in the recent years (see also [32, 33]).

In solving SFP, in p-uniformly convex and uniformly smooth real Banach spaces, it was proved that the $\{x_n\}$ defined by (1.1.2) converges weakly to a solution of SFP (1.1.1) provided the duality mapping J is weak-to-weak continuous and $t_n \in \left(0, (\frac{q}{C_q\|A\|^q})^{\frac{1}{q-1}}\right)$ where $\frac{1}{p}+\frac{1}{q}=1$ and C_q is the uniform smoothness coefficient of E_1. (See [26, 28]). Lately, Wang [30] modified the above algorithm (1.1.2) and proved strong convergence by using the idea in the work of Nakajo and Takahashi [21] in p-uniformly convex Banach spaces which is also uniformly smooth. The main advantage of result of Wang [30] is that the weak-to-weak continuity of the duality mapping, assumed in [24] is dispensed with and strong convergence result was achieved.

The class of left Bregman firmly nonexpansive mappings associated with the Bregman distance induced by a convex function was introduced and studied by Martin-Marques et al. [17]. If C is a nonempty and closed subset of int(dom f), where f is a Legendre and Fréchet differentiable function, and $T : C \to int\ (dom\ f)$ is a left Bregman strongly nonexpansive mapping, it is proved that $F(T)$ is closed (see [17]). In addition, they have shown that this class of mappings is closed under composition and convex combination and proved weak convergence of the Picard iterative method to a fixed point of a mapping under suitable conditions (see [16]). However, Picard iteration process has only $weak\ convergence$.

Recently, Shehu et al.[26] introduced an algorithm for solving split feasibility problems and fixed point problems such that the strong convergence is guaranteed by using Halpern's iteration process. Let $u \in E_1$ be fixed, $u_1 \in E_1$ arbitrarily. Let $\{x_n\}$ be the sequence generated by the following manner:

$$\begin{aligned} x_n &= \Pi_C J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))], \\ u_{n+1} &= \Pi_C J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n)),\ \ n \geq 1, \end{aligned} \tag{1.1.6}$$

where $\{\alpha_n\} \subset (0,1)$. It was proved that if $\alpha_n \to 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $t_n \in \left(0, (\frac{q}{C_q\|A\|^q})^{\frac{1}{q-1}}\right)$, then $\{x_n\}$ generated by (1.1.6) converges strongly to a solution of the SFP and fixed point of T which is a left Bregman strongly nonexpansive mappings.

In this paper, motivated by the works of López et al. [14] and Shehu et al. [26], we introduce a new self-adaptive method for solving the split feasibility problem and the fixed point problem of left Bregman strongly nonexpansive mappings in Banach spaces. We then prove its strong convergence of the sequence generated by our scheme in p-uniformly convex real Banach spaces which are also uniformly smooth. The advantage of our algorithm lies in the fact that step-sizes are dynamically chosen and not depend on the operator norm. Numerical experiments and some comparisons are included to show the effectiveness of the our algorithm. Our results mainly improve the results of Shehu et al. [26] and also complement many other results in the literature.

CHAPTER II

Preliminaries and lemmas

2.1 Preliminaries

In this section, we give some preliminaries which will be used in the sequel.

Definition 2.1.1 [35]**(Fixed point)**

Let X be a nonempty set and $T : X \to X$. We say that $x \in X$ is a fixed point of T if

$$T(x) = x$$

and denote by $Fix(T)$ the set of all fixed points of T.

Example 2.1.2 1. *If* $X = \mathbb{R}$ *and* $T(x) = x^2 + 5x + 4$*, then* $Fix(T) = \{-2\}$*;*

2. *If* $X = \mathbb{R}$ *and* $T(x) = x^2 - x$*, then* $Fix(T) = \{0, 2\}$*;*
3. *If* $X = \mathbb{R}$ *and* $T(x) = x + 5$*, then* $Fix(T) = \emptyset$*;*
4. *If* $X = \mathbb{R}$ *and* $T(x) = x$*, then* $Fix(T) = \mathbb{R}$*.*

Definition 2.1.3 [37]**(Normed space)**

Let X be a norm linear space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\|\cdot\| : X \to \mathbb{R}^+$ be a function. Then $\|\cdot\|$ is said to be a norm if the following properties hold:

1. $\|x\| \geq 0$, and $\|x\| = 0 \Leftrightarrow x = 0$;
2. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$;
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (triangle inequality).

The ordered pair $(X, \|\cdot\|)$ is called a *normed space*.

Example 2.1.4 *Let $X = \mathbb{R}^n$ is a normed space with the following norms :*

$$
\begin{aligned}
\|x\|_1 &= \sum_{i=1}^{n} |x_i| \; for\ all\ \ x = (x_1, x_2, .., x_n) \in \mathbb{R}^n; \\
\|x\|_p &= \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p} \; for\ all\ \ x = (x_1, x_2, .., x_n) \in \mathbb{R}^n \ and\ p \in (1, \infty); \\
\|x\|_\infty &= \max_{1 \le i \le n} |x_i| \; for\ all\ \ x = (x_1, x_2, .., x_n) \in \mathbb{R}^n.
\end{aligned}
$$

Definition 2.1.5 [37]**(Convergent sequence)**

A sequence $\{x_n\}$ in a normed space X is said to be convergent to x if $\lim_{n\to\infty} \|x_n - x\| = 0$. In this case, we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Definition 2.1.6 [37]**(Cauchy sequence)** A sequence $\{x_n\}$ in a normed space X is said to be *Cauchy* if $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$, *i.e.*, for $\epsilon > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n \ge n_0$.

Definition 2.1.7 [37]**(Completeness)**

The space X is said to be *complete* if every Cauchy sequence in X converges (that is, has a limit which is an element of X.)

Expressed in terms of completeness, the Cauchy convergence criterion implies the following.

Definition 2.1.8 [36]**(Banach space)**

A normed space which is complete with respect to the metric induced by the norm is called a Banach space.

Example 2.1.9 *The simplest example of a Banach space is $\mathbb{R}^N$ or $\mathbb{C}^N$ with the Euclidean norm.*

Definition 2.1.10 [37]**(Strong convergence)**

A sequence $\{x_n\}$ in a normed space X is said to be Strongly convergent (or convergent in the norm) if there is an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

Definition 2.1.11 [37]**(Inner product space)**

An inner product space is a vector space X with an inner product defined on X. Here, an inner product on X is a mapping of $X \times X$ into the scalar field $\mathbb{K}$ of X; that is, with every pair of vectors x and y there is associated a scalar which is written by $\langle x, y\rangle$ and called the *inner product* of x and y, such that for all vectors x, y, z and scalars α we have

(IP1) $\langle x, x\rangle \geq 0$;

(IP2) $\langle x, x\rangle = 0 \Leftrightarrow x = 0$;

(IP3) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$;

(IP4) $\langle x, y\rangle = \overline{\langle y, x\rangle}$;

(IP5) $\langle x + y, z\rangle = \langle x, z\rangle + \langle z, y\rangle$.

Proposition 2.1.12 [37]**(The Cauchy-Schwarz inequality)**

Let X be an inner product space. Then the following holds:

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle \text{ for all } x, y \in X, \tag{2.1.1}$$

i.e.,

$$|\langle x, y\rangle| \leq \|x\|\|y\| \text{ for all } x, y \in X. \tag{2.1.2}$$

Definition 2.1.13 [36]**(Hilbert space)**

An inner product space which is complete with respect to the induced norm is called a *Hilbert space*.

Definition 2.1.14 [37]**(Closed set)**

Let (X, d) be a metric space. A subset $U \subseteq X$ is called open if for every $x \in X$ there exists $r > 0$ such that $B(x, r) \subseteq U$. A set U is called closed if its complement $X \setminus U$ is open.

Definition 2.1.15 [37]**(Convex set)**

Let C be a subset of a linear space X. Then C is said to be *convex* if $(1-\lambda)x+\lambda y \in C$ for all $x, y \in C$ and all scalar $\lambda \in [0,1]$.

Definition 2.1.16 [34]**(Convex function)**

Let X be a linear space and $f : X \to (-\infty, \infty]$ be a function. Then f is said to be *convex* if $f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$ for all $x, y \in X$ and $\lambda \in [0,1]$.

Definition 2.1.17 [37]**(Bounded sequence)**

A sequence $\{x_n\}$ in X is bounded if there exists $M > 0$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Definition 2.1.18 [34]**(Bounded linear operator)**

Let X and Y be normed spaces and $T : X \to Y$ be a linear operator. The operator T is said to be *bounded* if there is a real number $c > 0$ such that for all $x \in X$,

$$\|Tx\| \leq c\|x\|.$$

2.2 Lemmas

Let E be a real Banach space with norm $\|\cdot\|$, and E^* denotes the Banach dual of E endowed with the dual norm $\|\cdot\|_*$. Let $1 < q \leq 2 \leq p$ with $\frac{1}{p}+\frac{1}{q}=1$. The *modulus of convexity* $\delta_E : [0,2] \to [0,1]$ is defined as

$$\delta_E(\epsilon) = inf\{1-\frac{\|x+y\|}{2} : \|x\| = 1 = \|y\|, \|x-y\| \geq \epsilon\}.$$

E is called *uniformly convex* if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0,2]$ and p-uniformly convex if there is a $C_p > 0$ such that $\delta_E(\epsilon) \geq C_p\epsilon^p$ for any $\epsilon \in (0,2]$. The *modulus of smoothness* $\rho_E(\tau) : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_E(\tau) = \{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1\}.$$

E is called *uniformly smooth* if $\lim\limits_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0$ and q-uniformly smooth if there is a $C_q > 0$ such that $\rho_E(\tau) \leq C_q\tau^q$ for any $\tau > 0$. The L_p space is 2-uniformly convex for $1 < p \leq 2$ and p-uniformly convex for $p \geq 2$. It is known that E is p-uniformly convex if and only if its dual E^* is q-uniformly smooth (see [13]).

The *duality mapping* J_E^p is one-to-one, single-valued and satisfies $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ the duality mapping of E^* (see [2, 11, 23]). Here the *duality mapping* $J_E^p : E \to 2^{E^*}$ defined by

$$J_E^p(x) = \{\overline{x} \in E^* : \langle x, \overline{x}\rangle = \|x\|^p, \|\overline{x}\| = \|x\|^{p-1}\}.$$

The duality mapping J_E^p is said to be weak-to-weak continuous if

$$x_n \rightharpoonup x \Rightarrow \langle J_E^p x_n, y\rangle \to \langle J_E^p x, y\rangle$$

holds true for any $y \in E$. It is worth noting that the $\ell_p (p > 1)$ space has such a property, but the $J_E^p (p > 2)$ space does not share this property.

Let $f : E \to \mathbb{R}$, the *Bregman distance* with respect to f is defined as:

$$\Delta_f(x,y) = f(y) - f(x) - \langle f'(x), y - x\rangle, \ \ x, y \in E$$

It is worth noting that the duality mapping J_p is in fact the derivative of the function $f_p(x) = \frac{1}{p}\|x\|^p$. Then the Bregman distance with respect to f_p is given by

$$\begin{aligned}\Delta_p(x,y) &= \frac{1}{q}\|x\|^p - \langle J_E^p x, y\rangle + \frac{1}{p}\|y\|^p \\ &= \frac{1}{p}(\|y\|^p - \|x\|^p) + \langle J_E^p x, x - y\rangle \\ &= \frac{1}{q}(\|x\|^p - \|y\|^p) - \langle J_E^p x - J_E^p y, x\rangle.\end{aligned}$$

We know the following inequality which was proved by Xu [31].

Lemma 2.2.1 [31] Let $x, y \in E$. If E is q-uniformly smooth, then there exists $C_q > 0$ such that

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_E^q(x)\rangle + C_q\|y\|^q.$$

Let $x, y, z \in E$, one can easily get

$$\Delta_p(x, y) + \Delta_p(y, z) - \Delta_p(x, z) = \langle x - y, J_E^p z - J_E^p y\rangle, \tag{2.2.1}$$

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_E^p x - J_E^p y\rangle \tag{2.2.2}$$

and

$$\Delta_p(x, y) = \frac{\|x\|^p}{p} + \frac{\|y\|^p}{q} - \langle x, J_E^p(y)\rangle, \tag{2.2.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

For the p-uniformly convex space, the metric and Bregman distance has the following relation (see [24]):

$$\tau\|x - y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J_E^p x - J_E^p y\rangle, \tag{2.2.4}$$

where $\tau > 0$ is some fixed number.

Proposition 2.2.2 [5, 12] Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that $\Delta_p(x_n, y_n) \to 0$. If $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.

Let C be a nonempty, closed and convex subset of E. The metric projection

$$P_C x = \text{argmin}_{y \in C}\|x - y\|, \quad x \in E,$$

is the unique minimizer of the norm distance, which can be characterized by a variational inequality:

$$\langle J_E^p(x - P_C x), z - P_C x\rangle \leq 0, \quad \forall z \in C. \tag{2.2.5}$$

Likewise, one can define the Bregman projection:

$$\Pi_C x = \text{argmin}_{y \in C} \Delta_p(x, y), \quad x \in E,$$

as the unique minimizer of the Bregman distance (see [25]). The Bregman projection can also be characterized by a variational inequality:

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x\rangle \leq 0, \quad \forall z \in C. \tag{2.2.6}$$

Moreover, we have

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x, z) - \Delta_p(x, \Pi_C x), \quad \forall z \in C. \tag{2.2.7}$$

Let E be a strictly convex, smooth and reflexive Banach space. Following [2, 9], we make use of the function $V_p : E^* \times E \to [0, +\infty)$, which is defined by

$$V_p(\overline{x}, x) = \tfrac{1}{q}\|\overline{x}\|^q - \langle \overline{x}, x\rangle + \tfrac{1}{p}\|x\|^p, \quad \forall x \in E, \ \overline{x} \in E^*,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then V_p is nonnegative and

$$V_p(\overline{x}, x) = \Delta_p(J_{E^*}^q(\overline{x}), x) \tag{2.2.8}$$

for all $x \in E$ and $\overline{x} \in E^*$. Moreover, using the subdifferential inequality for $f(x) = \frac{1}{q}\|x\|^q, \ x \in E^*$, we have

$$\langle J_E^q(x), y\rangle \leq \frac{1}{q}\|x + y\|^q - \frac{1}{q}\|x\|^q, \quad \forall x, y \in E^*. \tag{2.2.9}$$

Using (2.2.9), we have

$$V_p(\overline{x}, x) + \langle \overline{y}, J_{E^*}^q(\overline{x}) - x\rangle \leq V_p(\overline{x} + \overline{y}, x) \tag{2.2.10}$$

for all $x \in E$ and $\bar{x}, \bar{y} \in E^*$ (see, for example, [27, 29]). In addition, V_p is convex in the first variable since $\forall z \in E$,

$$\Delta_p\left(J_{E^*}^q\left(\sum_{i=1}^N t_i J_E^p(x_i)\right), z\right) = V_p\left(\sum_{i=1}^N t_i J_E^p(x_i), z\right) \leq \sum_{i=1}^N t_i \Delta_p(x_i, z), \tag{2.2.11}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Let C be a convex subset of int domf_p, where $f_p(x) = \frac{1}{p}\|x\|^p$, $2 \leq p < \infty$ and let T be a salf-mapping of C. A point $p \in C$ is said to be an *asymptotic fixed point* (please, see [10, 22]) of T if C contains a sequence $\{x_n\}_{n=1}^\infty$ which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

Definition 2.2.3 A nonlinear mapping T with a nonempty asymptotic fixed point set is said to be: (i) *left Bregman strongly nonexpansive* (L-BSNE) (see [16, 17]) with respect to a nonempty $\widehat{F}(T)$ if

$$\Delta_p(Tx, \bar{x}) \leq \Delta_p(x, \bar{x}),\ \forall x \in C,\ \bar{x} \in \widehat{F}(T)$$

and if whenever $\{x_n\} \subset C$ is bounded, $\bar{x} \in \widehat{F}(T)$ and

$$\lim_{n\to\infty} (\Delta_p(x_n, \bar{x}) - \Delta_p(Tx_n, \bar{x})) = 0,$$

it follows that

$$\lim_{n\to\infty} \Delta_p(x_n, Tx_n) = 0.$$

(ii) An operator $T : C \to E$ is said to be: *left Bregman firmly nonexpansive*

(L-BFNE) if

$$\langle J_p^E(Tx) - J_p^E(Ty), Tx - Ty\rangle \ \leq \langle J_p^E(Tx) - J_p^E(Ty), x - y\rangle$$

for any $x, y \in C$.

The class of left Bregman strongly nonexpansive mappings is of particular significance in fixed point, iteration and convex optimization theories mainly because it is closed under composition. For more information and examples of L-BSNE and L-BFNE operators. From [16, 17], we know that every left Bregman firmly nonexpansive mapping is left Bregman strongly nonexpansive if $F(T) = \widehat{F}(T)$.

We also need the following tools in analysis which will be used in the sequel.

Lemma 2.2.4 [15] Let $\{s_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ which satisfies $s_{n_i} < s_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:

$$\tau(n) = \max\{k \leq n : s_k < s_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : s_k < s_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq ...$ and $\tau(n) \to \infty$;

(ii) $s_{\tau(n)} \leq s_{\tau(n)+1}$ and $s_n \leq s_{\tau(n)+1}, \forall n \geq n_0$.

Lemma 2.2.5 [31] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation :*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n,\ n \geq 1,$$

where (i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\limsup_{n\to\infty} \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

CHAPTER III

Main results

3.1 Main theorem

In this section, we prove strong convergence theorem for the split feasibility problem in Banach spaces.

Theorem 3.1.1 Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and A^* : $E_2^* \to E_1^*$ be the adjoint of A. Let T be a left Bregman strongly nonexpansive mapping of C into it self such that $F(T) = \widehat{F}(T)$ and $F(T) \cap \Omega \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in (0,1). For a fixed $u \in E_1$, let sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be iteratively generated by $u_1 \in E_1$,

$$\begin{cases} x_n = \Pi_C J^q_{E_1^*}[J^p_{E_1}(u_n) - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \nabla f(u_n)], \\ u_{n+1} = \Pi_C J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n) J^p_{E_1}(Tx_n)), \quad n \geq 1, \end{cases} \tag{3.1.1}$$

where $f(u_n) = \frac{1}{p}\|(I - P_Q)Au_n\|^p$, $\nabla f(u_n) = A^* J^p_{E_2}(Au_n - P_Q(Au_n))$. *If* $\alpha_n \to 0$. $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ *and* $\{\rho_n\} \subset (0, \infty)$ *satisfies*

$$\inf_n \rho_n (pq - C_q \rho_n^{q-1}) > 0.$$

Then the sequence $\{u_n\}_{n=1}^{\infty}$ *converges strongly to an element* $x^* \in F(T) \cap \Omega$, *where* $x^* = \Pi_{F(T)\cap\Omega} u$.

Proof. We note that $\nabla f(u_n) = A^* J^p_{E_2}(Au_n - P_Q(Au_n))$ for all $n \in \mathbb{N}$. Set

$$y_n = J^p_{E_1}(u_n) - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \nabla f(u_n)$$

for all $n \in \mathbb{N}$. We see that $(p-1)q = p$. Then, by Lemma 2.2.1, we have

$$
\begin{aligned}
\|y_n\|^q &= \|J^p_{E_1}(u_n) - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \nabla f(u_n)||^q \\
&\leq ||u_n||^p - q\rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \langle u_n, \nabla f(u_n)\rangle + C_q \rho_n^q \frac{f^{(p-1)q}(u_n)}{||\nabla f(u_n)||^{pq}} ||\nabla f(u_n)||^q \\
&= ||u_n||^p - q\rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \langle u_n, \nabla f(u_n)\rangle + C_q \rho_n^q \frac{f^{p}(u_n)}{||\nabla f(u_n)||^{p}}. \qquad (3.1.2)
\end{aligned}
$$

Set $v_n = J^q_{E_1^*}[J^p_{E_1}(u_n) - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \nabla f(u_n)]$ for all $n \geq 1$. Then, we have $x_n = \Pi_C v_n$ for all $n \geq 1$. Let $x^* = \Pi_{F(T)\cap\Omega} u$. Then by (3.1.2), we have

$$
\begin{aligned}
\Delta_p(x_n, x^*) &\leq \Delta_p(v_n, x^*) = \Delta_p(J^q_{E_1^*}[J^p_{E_1}(u_n) - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \nabla f(u_n)], x^*) \\
&= \frac{||x^*||^p}{p} + \frac{1}{q} ||J^p_{E_1}(u_n) - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \nabla f(u_n)||^q - \langle J^p_{E_1}(u_n), x^*\rangle \\
&\quad + \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \langle x^*, \nabla f(u_n)\rangle \\
&\leq \frac{1}{q} ||u_n||^p - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \langle u_n, \nabla f(u_n)\rangle + \frac{C_q}{q} \rho_n^q \frac{f^p(u_n)}{||\nabla f(u_n)||^p} \\
&\quad - \langle x^*, J^p_{E_1}(u_n)\rangle + \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \langle x^*, \nabla f(u_n)\rangle + \frac{\|x^*\|^p}{p} \\
&= \frac{1}{q} ||u_n||^p - \langle x^*, J^p_{E_1}(u_n)\rangle + \frac{\|x^*\|^p}{p} + \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \langle x^* - u_n, \nabla f(u_n)\rangle \\
&\quad + \frac{C_q}{q} \rho_n^q \frac{f^p(u_n)}{||\nabla f(u_n)||^p} \\
&= \Delta_p(u_n, x^*) + \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \langle x^* - u_n, \nabla f(u_n)\rangle \\
&\quad + \frac{C_q}{q} \rho_n^q \frac{f^p(u_n)}{||\nabla f(u_n)||^p}. \qquad (3.1.3)
\end{aligned}
$$

On the other hand, we see that

$$
\begin{aligned}
\langle \nabla f(u_n), x^* - u_n\rangle &= \langle A^* J^p_{E_2}(Au_n - P_Q(Au_n)), x^* - u_n\rangle \\
&= \langle J^p_{E_2}(Au_n - P_Q(Au_n)), Ax^* - Au_n\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \langle J^p_{E_2}(Au_n - P_Q(Au_n)), P_Q(Au_n) - Au_n\rangle \\
&\quad + \langle J^p_{E_2}(Au_n - P_Q(Au_n)), Ax^* - P_Q(Au_n)\rangle \\
&\leq -\|Au_n - P_Q(Au_n)\|^p = -pf(u_n). \qquad (3.1.4)
\end{aligned}
$$

By (3.1.3) and (3.1.4), we obtain

$$
\begin{aligned}
\Delta_p(x_n, x^*) \leq \Delta_p(v_n, x^*) &\leq \Delta_p(u_n, x^*) + \frac{C_q}{q}\rho_n^q \frac{f^p(u_n)}{\|\nabla f(u_n)\|^p} - \rho_n \frac{pf^p(u_n)}{\|\nabla f(u_n)\|^p} \\
&= \Delta_p(u_n, x^*) + \left(\frac{C_q}{q}\rho_n^q - \rho_n p\right)\frac{f^p(u_n)}{\|\nabla f(u_n)\|^p}. \qquad (3.1.5)
\end{aligned}
$$

Since $\inf_n \rho_n(pq - C_q\rho_n^{q-1}) > 0$, we have

$$
\Delta_p(x_n, x^*) \leq \Delta_p(u_n, x^*), \quad \forall n \geq 1.
$$

Now using (3.1.1), we have

$$
\begin{aligned}
\Delta_p(x_{n+1}, x^*) \leq \Delta_p(u_{n+1}, x^*) &\leq \Delta_p(J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n)), x^*) \\
&\leq \alpha_n\Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(Tx_n, x^*) \\
&\leq \alpha_n\Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(x_n, x^*) \qquad (3.1.6) \\
&\leq \max\{\Delta_p(u, x^*), \Delta_p(x_n, x^*)\} \\
&\ \ \vdots \\
&\leq \max\{\Delta_p(u, x^*), \Delta_p(x_1, x^*)\}.
\end{aligned}
$$

Hence $\{u_n\}_{n=1}^\infty$ is bounded. Also $\{x_n\}_{n=1}^\infty$ is bounded.

Let $b_n := J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n)), \ n \geq 1$. Then we obtain

$$
\begin{aligned}
\Delta_p(b_n, Tx_n) &\leq \alpha_n\Delta_p(u, Tx_n) + (1-\alpha_n)\Delta_p(Tx_n, Tx_n) \\
&= \alpha_n\Delta_p(u, Tx_n) \to 0, \quad n \to \infty.
\end{aligned}
$$

Set $w_n = \alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n)$ for all $n \geq 1$. We next consider the following estimation:

$$
\begin{aligned}
\Delta_p(x_{n+1}, x^*) &\leq \Delta_p(u_{n+1}, x^*) = \Delta_p(\Pi_C b_n, x^*) \leq \Delta_p(b_n, x^*) - \Delta_p(b_n, \Pi_C b_n) \\
&= \Delta_p(J^q_{E_1^*}[\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n)], x^*) - \Delta_p(b_n, \Pi_C b_n) \\
&= V_p(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n), x^*) - \Delta_p(b_n, \Pi_C b_n) \\
&\leq V_p(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n) - \alpha_n(J^p_{E_1}(u) - J^p_{E_1}(x^*)), x^*) \\
&\quad +\alpha_n\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), J^q_{E_1^*}(w_n) - x^*\rangle - \Delta_p(b_n, \Pi_C b_n) \\
&= V_p(\alpha_n J^p_{E_1}(x^*) + (1-\alpha_n)J^p_{E_1}(Tx_n), x^*) \\
&\quad +\alpha_n\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), J^q_{E_1^*}(w_n) - x^*\rangle - \Delta_p(b_n, \Pi_C b_n) \\
&\leq (1-\alpha_n)V_p(J^p_{E_1}(Tx_n), x^*) \\
&\quad +\alpha_n\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), J^q_{E_1^*}(w_n) - x^*\rangle - \Delta_p(b_n, \Pi_C b_n) \\
&= (1-\alpha_n)\Delta_p(Tx_n, x^*) + \alpha_n\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), J^q_{E_1^*}(w_n) - x^*\rangle \\
&\quad -\Delta_p(b_n, \Pi_C b_n) \\
&\leq (1-\alpha_n)\Delta_p(x_n, x^*) + \alpha_n\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), J^q_{E_1^*}(w_n) - x^*\rangle \\
&\quad -\Delta_p(b_n, \Pi_C b_n). \qquad (3.1.7)
\end{aligned}
$$

Let $s_n = \Delta_p(x_n, x^*)$ $\forall n \in \mathbb{N}$. Then, by (3.1.7), we have

$$
\begin{aligned}
s_{n+1} &\leq (1-\alpha_n)s_n + \alpha_n\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), J^q_{E_1^*}(w_n) - x^*\rangle \\
&\quad -\Delta_p(b_n, \Pi_C b_n). \qquad (3.1.8)
\end{aligned}
$$

We next consider the following two cases:

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\Delta_p(x_n, x^*)\}_{n=n_0}^{\infty}$ is non-increasing. Then $\{\Delta_p(x_n, x^*)\}_{n=1}^{\infty}$ converges and $\Delta_p(x_n, x^*) - \Delta_p(x_{n+1}, x^*) \to 0,\ n \to \infty$. Now, from (3.1.5), we obtain

$$
(\rho_n p - \frac{C_q}{q}\rho_n^q)\frac{f^p(u_n)}{\|\nabla f(u_n)\|^p} \leq \Delta_p(u_n, x^*) - \Delta_p(x_n, x^*). \qquad (3.1.9)
$$

Also, from (3.1.6), we have

$$\Delta_p(u_{n+1}, x^*) \leq \alpha_n \Delta_p(u, x^*) + \Delta_p(x_n, x^*). \tag{3.1.10}$$

Putting (3.1.10) into (3.1.9), we have

$$\begin{aligned}(\rho_n p - \frac{C_q}{q}\rho_n^q)\frac{f^p(u_n)}{\|\nabla f(u_n)\|^p} &\leq \Delta_p(u_n, x^*) - \Delta_p(x_n, x^*) \\ &\leq \alpha_{n-1}\Delta_p(u, x^*) + \Delta_p(x_{n-1}, x^*) \\ &\quad -\Delta_p(x_n, x^*). \end{aligned} \tag{3.1.11}$$

By $\inf\limits_n \rho_n(pq - C_q\rho_n^{q-1}) > 0$ and (3.1.11), we have

$$0 < (\rho_n p - \tfrac{C_q}{q}\rho_n^q)\tfrac{f^p(u_n)}{\|\nabla f(u_n)\|^p} \leq \alpha_{n-1}\Delta_p(u, x^*) + \Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*) \to 0$$

as $n \to \infty$. It follows that $f(u_n) \to 0,\ n \to \infty$, since $\{\|\nabla f(u_n)\|\}$ is bounded. Hence

$$\lim_{n\to\infty} \|Au_n - P_Q(Au_n)\| = 0. \tag{3.1.12}$$

From (3.1.8), we have

$$\begin{aligned} 0 &\leq \Delta_p(b_n, \Pi_C b_n) \\ &\leq (s_n - s_{n+1}) + \alpha_n[\langle J_{E_1}^p(u) - J_{E_1}^p(x^*), J_{E_1^*}^q(w_n) - x^*\rangle - s_n] \to 0,\ n \to \infty. \end{aligned}$$

Hence, by Proposition 2.2.2, we obtain

$$\|b_n - \Pi_C b_n\| \to 0,\ n \to \infty. \tag{3.1.13}$$

It also follows that

$$\begin{aligned} 0 \leq \|J_{E_1}^p(v_n) - J_{E_1}^p(u_n)\| &= \|J_{E_1}^p(u_n) - \rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\nabla f(u_n) - J_{E_1}^p(u_n)\| \\ &= \|\rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\nabla f(u_n)\| \to 0,\ n \to \infty. \end{aligned}$$

Therefore, we obtain

$$\lim_{n\to\infty} \|J^p_{E_1}(v_n) - J^p_{E_1}(u_n)\| = 0.$$

Since $J^q_{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} \|v_n - u_n\| = 0.$$

Furthermore, we have from (2.2.7), (3.1.5) and (3.1.6) that

$$\begin{aligned}\Delta_p(v_n, x_n) = \Delta_p(v_n, \Pi_C v_n) &\leq \Delta_p(v_n, x^*) - \Delta_p(x_n, x^*)\\ &\leq \Delta_p(u_n, x^*) - \Delta_p(x_n, x^*)\\ &\leq \alpha_{n-1}M^* + \Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*) \to 0,\ n\to\infty,\end{aligned}$$

for some $M^* > 0$. By Proposition 2.2.2, we have that $\|v_n - x_n\| \to 0,\ n \to \infty$. Hence,

$$\|x_n - u_n\| \leq \|x_n - v_n\| + \|v_n - u_n\| \to 0,\ n\to\infty.$$

Observe that $\Delta_p(x_{n+1}, x^*) \leq \Delta_p(u_{n+1}, x^*) \leq \alpha_n \Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(Tx_n, x^*)$. It then follows that

$$\begin{aligned}\Delta_p(x_n, x^*) - \Delta_p(Tx_n, x^*) &= \Delta_p(x_n, x^*) - \Delta_p(x_{n+1}, x^*)\\ &\quad +\Delta_p(x_{n+1}, x^*) - \Delta_p(Tx_n, x^*)\\ &\leq \Delta_p(x_n, x^*) - \Delta_p(x_{n+1}, x^*)\\ &\quad +\alpha_n(\Delta_p(u, x^*) - \Delta_p(Tx_n, x^*))\\ &\to 0,\ n\to\infty.\end{aligned}$$

Then we obtain

$$\lim_{n\to\infty} \Delta_p(x_n, Tx_n) = 0.$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to z. Now, since $x_{n_j} \rightharpoonup z$ and $\lim\limits_{n\to\infty} \|x_n - u_n\| = 0$, we obtain a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ that $u_{n_j} \rightharpoonup z$. Since $F(T) = \widehat{F}(T)$, we have $z \in F(T)$.

Next, we show that $z \in \Omega$. From (2.2.2), (2.2.4) and (2.2.6), we have

$$\begin{aligned}
\Delta_p(z, \Pi_C z) &\leq \langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), z - \Pi_C z\rangle \\
&= \langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), z - u_{n_j}\rangle \\
&\quad + \langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), u_{n_j} - \Pi_C u_{n_j}\rangle \\
&\quad + \langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), \Pi_C u_{n_j} - \Pi_C z\rangle \\
&\leq \langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), z - u_{n_j}\rangle \\
&\quad + \langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), u_{n_j} - \Pi_C u_{n_j}\rangle \\
&\to 0,
\end{aligned}$$

as $j \to \infty$. So we have $\Delta_p(z, \Pi_C z) = 0$. Thus, $z \in C$. Let us now fix $x \in C$ such that $Ax \in Q$. Then

$$\begin{aligned}
\|Au_{n_j} - P_Q(Au_{n_j})\|^p &= \langle J_{E_2}^p(Au_{n_j} - P_Q(Au_{n_j})), Au_{n_j} - P_Q(Au_{n_j})\rangle \\
&= \langle J_{E_2}^p(Au_{n_j} - P_Q(Au_{n_j})), Au_{n_j} - Ax)\rangle \\
&\quad + \langle J_{E_2}^p(Au_{n_j} - P_Q(Au_{n_j}), Ax - P_Q(Au_{n_j}))\rangle \\
&\leq \langle J_{E_2}^p(Au_{n_j} - P_Q(Au_{n_j})), Au_{n_j} - Ax)\rangle \\
&\leq M\|A^*(I - P_Q)Au_{n_j}\|^{p-1} \\
&\to 0, \; n \to \infty.
\end{aligned}$$

where $M > 0$ is sufficiently large number. It then follows from (2.2.5) that

$$\begin{aligned}
\|Az - P_Q(Az)\|^p &= \langle J^p_{E_2}(Az - P_Q(Az)), Az - P_Q(Az)\rangle \\
&= \langle J^p_{E_2}(Az - P_Q(Az)), Az - Au_{n_j}\rangle \\
&\quad + \langle J^p_{E_2}(Az - P_Q(Az)), Au_{n_j} - P_Q(Au_{n_j})\rangle \\
&\quad + \langle J^p_{E_2}(Az - P_Q(Az), P_Q(Au_{n_j}) - P_Q(Az)\rangle \\
&\leq \langle J^p_{E_2}(Az - P_Q(Az)), Az - Au_{n_j}\rangle \\
&\quad + \langle J^p_{E_2}(Az - P_Q(Az)), Au_{n_j} - P_Q(Au_{n_j})\rangle.
\end{aligned}$$

Since $u_{n_j} \rightharpoonup z,\ Au_{n_j} \rightharpoonup Az$ and $\|Au_{n_j} - P_Q(Au_{n_j})\| \to 0,\ j \to \infty$, it follows that

$$\|Az - P_Q(Az)\| = 0.$$

Hence, $Az \in Q$. This shows that $z \in \Omega$ and therefore $z \in F(T) \cap \Omega$.

Moreover, we see that

$$\Delta_p(x_n, b_n) \leq \alpha_n \Delta_p(x_n, u) + (1 - \alpha_n)\Delta_p(x_n, Tx_n) \to 0,\ \ n \to \infty.$$

It follows that $\|x_n - b_n\| \to 0,\ n \to \infty$. We next show that

$$\limsup_{n\to\infty} \langle J^p_{E_1}(u) - J^p_{E_1}(x^*), b_n - x^*\rangle \leq 0.$$

We choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle J^p_{E_1}(u) - J^p_{E_1}(x^*), x_n - x^*\rangle = \lim_{j\to\infty} \langle J^p_{E_1}(u) - J^p_{E_1}(x^*), x_{n_j} - x^*\rangle.$$

From $\|x_n - b_n\| \to 0, n \to \infty$ and (2.2.6), we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \langle J^p_{E_1}(u) - J^p_{E_1}(x^*), b_n - x^*\rangle &= \limsup_{n\to\infty} \langle J^p_{E_1}(u) - J^p_{E_1}(x^*), x_n - x^*\rangle \\
&= \langle J^p_{E_1}(u) - J^p_{E_1}(x^*), z - x^*\rangle \\
&\leq 0. \qquad (3.1.14)
\end{aligned}$$

Note that $\|J^p_{E_1}(Tx_n) - w_n\| = \alpha_n\|J^p_{E_1}(Tx_n) - J^p_{E_1}(u)\| \to 0,\ n \to \infty$. On the other hand, we see that

$$\begin{aligned}\|J^p_{E_1}(b_n) - w_n\| &= \|\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n) - w_n\| \\ &\leq \alpha_n\|J^p_{E_1}(u) - w_n\| + \|J^p_{E_1}(Tx_n) - w_n\| \to 0,\ n \to \infty.\end{aligned}$$

This shows that $\|b_n - J^q_{E_1^*}(w_n)\| \to 0,\ n \to \infty$. So we obtain by (3.1.14)

$$\limsup_{n\to\infty}\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), J^q_{E_1^*}(w_n) - x^*\rangle \leq 0. \tag{3.1.15}$$

Now, using (3.1.8), (3.1.15) and Lemma 2.2.5, we obtain $\Delta_p(x_n, x^*) \to 0,\ n \to \infty$. Hence, $x_n \to x^*$ as $n \to \infty$. Also we have $\|u_n - x^*\| \leq \|u_n - x_n\| + \|x_n - x^*\| \to 0,\ n \to \infty$. Thus $u_n \to x^*\ as\ n \to \infty$.

$Case$ 2: Assume that $\{s_n\}$ is not monotonically decreasing sequence, and let $\tau : \mathbb{N} \to \mathbb{N}$ be as in Lemma 2.2.4. We see that, by Lemma 2.2.4 (ii)

$$\begin{aligned}\Delta_p(x_{\tau(n)}, x^*) - \Delta_p(Tx_{\tau(n)}, x^*) &= \Delta_p(x_{\tau(n)}, x^*) - \Delta_p(x_{\tau(n)+1}, x^*) \\ &\quad + \Delta_p(x_{\tau(n)+1}, x^*) - \Delta_p(Tx_{\tau(n)}, x^*) \\ &\leq \alpha_n(\Delta_p(u, x^*) - \Delta_p(Tx_{\tau(n)}, x^*)) \\ &\to 0,\ n \to \infty.\end{aligned}$$

It then follows that

$$\lim_{n\to\infty}\Delta_p(x_{\tau(n)}, Tx_{\tau(n)}) = 0.$$

Similar to Case 1, we can show that $\|Au_{\tau(n)} - P_Q Au_{\tau(n)}\| \to 0,\ n \to \infty$ and

$$\limsup_{n\to\infty}\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), J^q_{E_1^*}(w_{\tau(n)}) - x^*\rangle \leq 0.$$

Also from (3.1.8), we have that

$$s_{\tau(n)+1} \leq (1-\alpha_{\tau(n)})s_{\tau(n)} + \alpha_{\tau(n)}\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), J^q_{E_1^*}(w_{\tau(n)}) - x^*\rangle,$$

which gives

$$s_{\tau(n)} \leq \langle J_{E_1}^p(u) - J_{E_1}^p(x^*), J_{E_1^*}^q(w_{\tau(n)}) - x^* \rangle.$$

So by Lemma 2.2.5, we obtain

$$\lim_{n\to\infty} s_{\tau(n)} = 0.$$

We next show that $\lim_{n\to\infty} s_{\tau(n)+1} = 0$. To show this, it suffices to prove that $\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0,\ n \to \infty$. Indeed, by (3.1.13), we observe that

$$\begin{aligned} \|x_{\tau(n)} - u_{\tau(n)+1}\| &\leqslant \|x_{\tau(n)} - b_{\tau(n)}\| + \|b_{\tau(n)} - \Pi_C b_{\tau(n)}\| + \|\Pi_C b_{\tau(n)} - u_{\tau(n)+1}\| \\ &\to 0, \quad n \to \infty. \end{aligned}$$

This shows that

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \leq \|x_{\tau(n)+1} - u_{\tau(n)+1}\| + \|u_{\tau(n)+1} - x_{\tau(n)}\| \to 0,\ n \to \infty.$$

From (2.2.1), it follows that

$$\begin{aligned} &\Delta_p(x^*, x_{\tau(n)+1}) + \Delta_p(x_{\tau(n)+1}, x_{\tau(n)}) - \Delta_p(x^*, x_{\tau(n)}) \\ &= \langle x^* - x_{\tau(n)+1}, J_{E_1}^p(x_{\tau(n)}) - J_{E_1}^p(x_{\tau(n)+1}) \rangle. \end{aligned}$$

Hence

$$\begin{aligned} s_{\tau(n)+1} &= \Delta_p(x^*, x_{\tau(n)+1}) \\ &\leq \Delta_p(x^*, x_{\tau(n)}) + \langle x^* - x_{\tau(n)+1}, J_{E_1}^p(x_{\tau(n)}) - J_{E_1}^p(x_{\tau(n)+1}) \rangle \to 0. \end{aligned}$$

Thus, by Lemma 2.2.4, we obtain $0 \leq s_n \leq s_{\tau(n)+1}$, which implies that $\lim\limits_{n\to\infty} s_n = 0$. This shows that $x_n \to x^*$ as $n \to \infty$, and hence $u_n \to x^*$ as $n \to \infty$. We thus complete the proof. □

CHAPTER IV

Numerical Examples

In this section, we provide some numerical examples and illustrate its performance by using Algorithm (3.1.1). Firstly, numerical results are shown in different choices of the step-size ρ_n with different values u and u_1.

Example 4.1 Let $E_1 = E_2 = L_2([0,1])$ with the inner product given by

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

Let

$$C := \{x \in L_2([0,1]) : \|x\|_{L_2} \leq 1\}.$$

Then

$$\Pi_C(x) = P_C(x) = \begin{cases} x, & \|x\| \leq 1 \\ \frac{x}{\|x\|}, & \|x\| > 1. \end{cases}$$

Also, let

$$Q := \{x \in L_2([0,1]) : \langle x, a \rangle = b\},$$

where $a = \frac{t}{2}, \quad b = 0$. Then

$$P_Q(x) = \frac{b - \langle a, x \rangle}{\|a\|_2^2} a + x.$$

Let us assume that $A : L_2([0,1]) \to L_2([0,1]), \quad (Ax)(t) = \frac{x(t)}{2}$. Then A is a bounded linear operator and $A^* = A$. Suppose that we take operator T in Theorem 3.1.1 as $T := P_C$, the metric projection on C (please see [16, 17]). Take $\alpha_n = \frac{1}{n+1}, \quad \forall n \geq 1$, then our iterative scheme (3.1.1) becomes

$$\begin{aligned} x_n &= P_C[u_n - \rho_n \frac{f(u_n)}{\|\nabla f(u_n)\|^2} A^*(Au_n - P_Q(Au_n))] \\ u_{n+1} &= P_C[\frac{u}{n+1} + (1 - \frac{1}{n+1})(P_C x_n)], \quad n \geq 1, \end{aligned} \tag{4.1}$$

where $f(u_n) = \frac{1}{2}\|Au_n - P_Q(Au_n)\|^2$ and $\nabla f(u_n) = A^*(Au_n - P_Q(Au_n))$ for all $n \in \mathbb{N}$.

We now study the effect (in terms of convergence, number of iterations required and the cpu time) of the sequence $\{\rho_n\} \subset (0, \infty)$ on the iterative scheme by choosing different ρ_n such that $\inf_n \rho_n(4 - \rho_n) > 0$ in the following cases.

Case 1: $\rho_n = \frac{0.5n}{n+1}$;

Case 2: $\rho_n = \frac{n}{n+1}$;

Case 3: $\rho_n = \frac{2n}{n+1}$;

Case 4: $\rho_n = \frac{3.5n}{n+1}$.

The stopping criterion is defined by $E_n = \frac{1}{2}\|Au_n - P_Q(Au_n)\|_{L_2}^2 < 10^{-3}$, or using stopping criterion $n = 1,000$. We choose different choices of u and u_1 as

Choice 1: $u = t$ and $u_1 = \sin(t) + t^2$;

Choice 2: $u = t^2$ and $u_1 = e^t + 2t$.

The numerical experiments, using our Algorithm (3.1.1), for each case and choice are reported in the following Table 4.1.

Table 4.1: Algorithm (3.1.1) with different cases of ρ_n and different choices of u and u_1

		Choice 1	Choice 2
Case 1	No. of Iter.	26	20
	cpu (Time)	1.247811	0.950551
Case 2	No. of Iter.	14	10
	cpu (Time)	0.647647	0.467636
Case 3	No. of Iter.	7	5
	cpu (Time)	0.327002	0.235971
Case 4	No. of Iter.	4	3
	cpu (Time)	0.191387	0.143973

The error plotting of E_n for each choice of u and u_1 is shown in Figure 1-2, respectively.

Figure 1: Different Cases with Choice 1

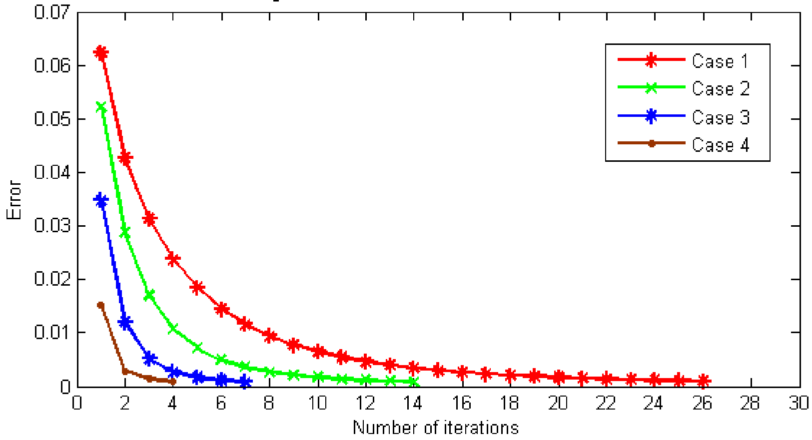
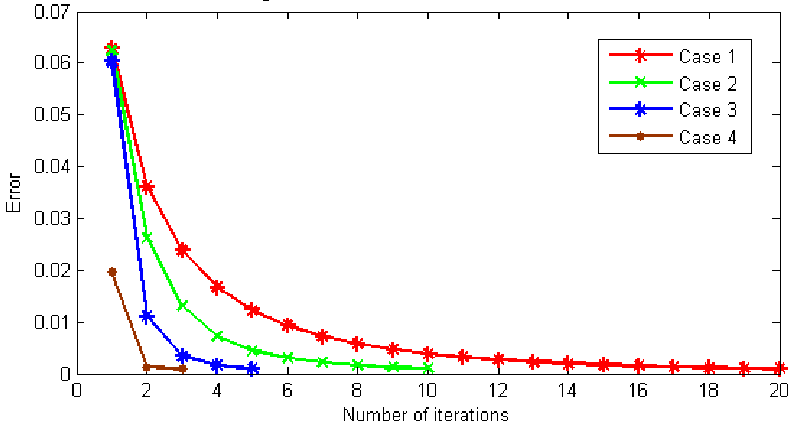


Figure 2: Different Cases with Choice 2



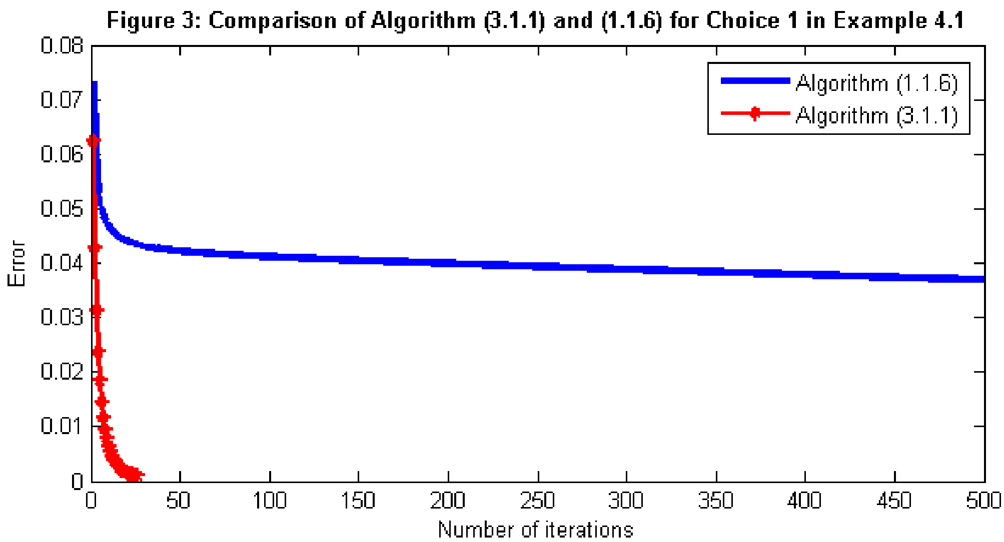
Remark 4.0.2. From our numerical experiments, it is observed that the different choices of u and u_1 has no effect in terms of cpu run time for the convergence of our algorithm. It is observed that the number of iterations and the cpu run time are significantly decreasing starting from Case 1 to Case 4.

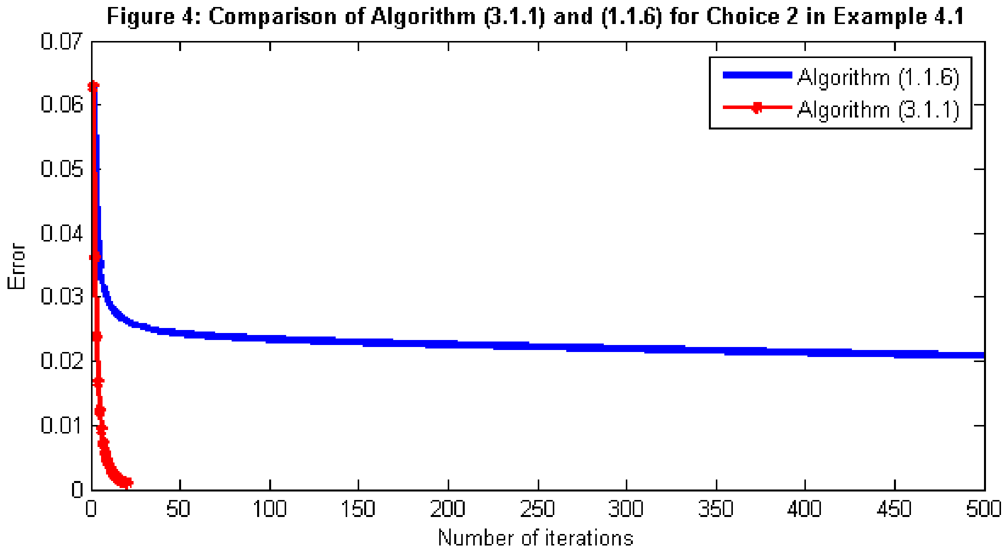
Finally, we comparison of convergence of Algorithm (3.1.1) and Algorithm (1.1.6). Let $\alpha_n = \frac{1}{n+1}$, for algorithm (3.1.1), we take $\rho_n = \frac{0.5n}{n+1}$ and for algorithm (1.1.6), we take $t_n = 0.001$. We use stopping criterion $n = 1,000$. For points u and u_1 randomly, we obtain the following numerical results.

Table 4.2: Comparison of Algorithm (3.1.1) and Algorithm (1.1.6) in Example 4.1

		Algorithm (3.1.1)	Algorithm (1.1.6)
Choice 1	No. of Iter.	26	$>$ 1,000
	cpu (Time)	1.247811	-
Choice 2	No. of Iter.	20	$>$ 1,000
	cpu (Time)	0.950551	-

The error plotting $n = 1,000$ of Algorithm (3.1.1) and Algorithm (1.1.6) for each choice is shown in Figure 3-4, respectively.





Remark 4.0.3. In numerical experiment, it is revealed that the sequence generated by our proposed Algorithm (3.1.1) using the self-adaptive technique converges more quickly than by Algorithm (1.1.6) of Shehu et al. [26] does.

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APPENDIX

A self-adaptive method for solving the split feasibility problem and the fixed point problem of Bregman strongly nonexpansive mappings in Banach spaces

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Abstract

In this work, we suggest a new self-adaptive method for finding a common solution of the split feasibility problem and the fixed point problem of Bregman strongly nonexpansive mappings. We prove its strong convergence theorem under some mild conditions. We also give some numerical examples to show the efficiency and implementation of our method.

Keywords: split feasibility problem; strong convergence; self-adaptive method; uniformly convex; uniformly smooth; fixed point problem; left Bregman strongly nonexpansive mappings; Banach space.

AMS Subject Classification: 49J53, 65K10, 49M37, 90C25.

1 Introduction

Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively; Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A which is defined by

$$\langle A^*\bar{y}, x\rangle := \langle \bar{y}, Ax\rangle,\ \forall x \in E_1, \bar{y} \in E_2^*.$$

The *split feasibility problem* (SFP) is to find a point

$$x \in C \ \text{ such that } Ax \in Q. \tag{1.1}$$

We denote by $\Omega = C \cap A^{-1}(Q) = \{y \in C : Ay \in Q\}$ the solution set of SFP. Then we have that Ω is a closed and convex subset of E_1.

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The SFP in finite-dimensional Hilbert spaces was introduced by Censor and Elfving [8] for modelling inverse problems which arise from phase retrievals, medical image reconstruction and recently in modelling of intensity modulated radiation therapy. The SFP attracts the attention of many authors due to its application in signal processing. Various algorithms and some interesting results have been invented to solve it (see, for example, [1, 3, 4, 6, 14, 18, 19, 20, 30]).

For solving SFP, in p-uniformly convex and uniformly smooth real Banach spaces, Schöpfer et al [24] proposed the following algorithm: For $x_1 \in E_1$ and

$$x_{n+1} = \Pi_C J_{E_1}^*[J_{E_1}(x_n) - t_n A^* J_{E_2}(Ax_n - P_Q(Ax_n))], \quad n \geq 1, \tag{1.2}$$

where Π_C denotes the Bregman projection and J the duality mapping. Clearly, the above algorithm covers the CQ-algorithm which was introduced by Byrne [7], which is defined by

$$x_{n+1} = P_C(x_n - \mu_n A^*(I - P_Q)Ax_n), \quad n \geq 1, \tag{1.3}$$

where $\mu_n \in (0, \frac{2}{\|A\|^2})$ and P_C, P_Q are the metric projections on C and Q, respectively, which is found to be a gradient-projection method in convex minimization as a special case. It was proved that $\{x_n\}$ defined by (1.3) converges weakly to a solution of SFP.

We observe that the operator norm $\|A\|$ may not be calculated easily in general. To overcome this difficulty, López et al. [14] suggested the following self-adaptive method, which permits step-size μ_n being selected self-adaptively in such a way:

$$\mu_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \quad n \geq 1, \tag{1.4}$$

where $\rho_n \in (0, 4)$, $f(x_n) = \frac{1}{2}\|(I - P_Q)Ax_n\|^2$ and $\nabla f(x_n) = A^*(I - P_Q)Ax_n$ for all $n \geq 1$. It was proved that the sequence $\{x_n\}$ defined by (1.4) converges weakly to a solution of SFP.

Also, employing the idea of Halpern's iteration, López et al. [14] proposed the following iteration method:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)P_C(x_n - \mu_n \nabla f(x_n)), \quad n \geq 1, \tag{1.5}$$

where $\{\alpha_n\} \subset [0, 1]$, $u \in C$ and the step-size μ_n is chosen as above. It was proved that $\{x_n\}$ defined by (1.5) converges strongly to a solution of SFP provided $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty} \alpha_n = \infty$. After that, there have been many modifications of the CQ algorithm and the self-adaptive method established in the recent years (see also [32, 33]).

In solving SFP, in p-uniformly convex and uniformly smooth real Banach spaces, it was proved that the $\{x_n\}$ defined by (1.2) converges weakly to a solution of SFP (1.1) provided the duality mapping J is weak-to-weak continuous and $t_n \in \left(0, (\frac{q}{C_q\|A\|^q})^{\frac{1}{q-1}}\right)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and C_q is the uniform smoothness coefficient of E_1. (See [26, 28]). Lately, Wang [30] modified the above algorithm (1.2) and proved strong convergence by using the idea in the work of Nakajo and Takahashi [21] in p-uniformly convex Banach spaces which is also uniformly smooth. The main advantage of result of Wang [30] is that the weak-to-weak continuity of the duality mapping, assumed in [24] is dispensed with and strong convergence result was achieved.

The class of left Bregman firmly nonexpansive mappings associated with the Bregman distance induced by a convex function was introduced and studied by Martin-Marques et al. [17]. If C is a nonempty and closed subset of int(dom f), where f is a Legendre and Fréchet differentiable function, and $T : C \to int\ (dom\ f)$ is a left Bregman strongly nonexpansive mapping, it is proved that $F(T)$ is closed (see [17]). In addition, they have shown that this class of mappings is closed under composition and convex combination and proved weak convergence of the Picard iterative method to a fixed point of a mapping under suitable conditions (see [16]). However, Picard iteration process has only *weak convergence*.

Recently, Shehu et al.[26] introduced an algorithm for solving split feasibility problems and fixed point problems such that the strong convergence is guaranteed by using Halpern's iteration process. Let $u \in E_1$ be fixed, $u_1 \in E_1$ arbitrarily. Let $\{x_n\}$ be the sequence generated by the following manner:

$$\begin{aligned} x_n &= \Pi_C J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))], \\ u_{n+1} &= \Pi_C J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n) J^p_{E_1}(Tx_n)), \ \ n \geq 1, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\} \subset (0,1)$. It was proved that if $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $t_n \in \Big(0, (\frac{q}{C_q\|A\|^q})^{\frac{1}{q-1}}\Big)$, then $\{x_n\}$ generated by (1.6) converges strongly to a solution of the SFP and fixed point of T which is a left Bregman strongly nonexpansive mappings.

In this paper, motivated by the works of López et al. [14] and Shehu et al. [26], we introduce a new self-adaptive method for solving the split feasibility problem and the fixed point problem of left Bregman strongly nonexpansive mappings in Banach spaces. We then prove its strong convergence of the sequence generated by our scheme in p-uniformly convex real Banach spaces which are also uniformly smooth. The advantage of our algorithm lies in the fact that step-sizes are dynamically chosen and not depend on the operator norm. Numerical experiments and some comparisons are included to show the effectiveness of the our algorithm. Our results mainly improve the results of Shehu et al. [26] and also complement many other results in the literature.

2 Preliminaries and lemmas

Let E be a real Banach space with norm $\|\cdot\|$, and E^* denotes the Banach dual of E endowed with the dual norm $\|\cdot\|_*$. Let $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. The *modulus of convexity* $\delta_E : [0,2] \to [0,1]$ is defined as

$$\delta_E(\epsilon) = inf\{1 - \frac{\|x+y\|}{2} : \|x\| = 1 = \|y\|, \|x-y\| \geq \epsilon\}.$$

E is called *uniformly convex* if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0,2]$ and p-uniformly convex if there is a $C_p > 0$ such that $\delta_E(\epsilon) \geq C_p \epsilon^p$ for any $\epsilon \in (0,2]$. The *modulus of smoothness* $\rho_E(\tau) : [0,\infty) \to [0,\infty)$ is defined by

$$\rho_E(\tau) = \{\frac{\| x + \tau y \| + \| x - \tau y \|}{2} - 1 : \|x\| = \|y\| = 1\}.$$

E is called *uniformly smooth* if $\lim\limits_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$ and q-uniformly smooth if there is a $C_q > 0$ such that $\rho_E(\tau) \leq C_q \tau^q$ for any $\tau > 0$. The L_p space is 2-uniformly convex for $1 < p \leq 2$ and p-uniformly convex for $p \geq 2$. It is known that E is p-uniformly convex if and only if its dual E^* is q-uniformly smooth (see [13]).

The *duality mapping* J_E^p is one-to-one, single-valued and satisfies $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ the duality mapping of E^* (see [2, 11, 23]). Here the *duality mapping* $J_E^p : E \to 2^{E^*}$ defined by

$$J_E^p(x) = \{\overline{x} \in E^* : \langle x, \overline{x} \rangle = \|x\|^p, \|\overline{x}\| = \|x\|^{p-1}\}.$$

The duality mapping J_E^p is said to be weak-to-weak continuous if

$$x_n \rightharpoonup x \Rightarrow \langle J_E^p x_n, y \rangle \to \langle J_E^p x, y \rangle$$

holds true for any $y \in E$. It is worth noting that the $\ell_p (p > 1)$ space has such a property, but the $J_E^p (p > 2)$ space does not share this property.

Let $f : E \to \mathbb{R}$, the *Bregman distance* with respect to f is defined as:

$$\Delta_f(x, y) = f(y) - f(x) - \langle f'(x), y - x \rangle, \;\; x, y \in E$$

It is worth noting that the duality mapping J_p is in fact the derivative of the function $f_p(x) = \frac{1}{p}\|x\|^p$. Then the Bregman distance with respect to f_p is given by

$$\begin{aligned} \Delta_p(x, y) &= \frac{1}{q}\|x\|^p - \langle J_E^p x, y \rangle + \frac{1}{p}\|y\|^p \\ &= \frac{1}{p}(\|y\|^p - \|x\|^p) + \langle J_E^p x, x - y \rangle \\ &= \frac{1}{q}(\|x\|^p - \|y\|^p) - \langle J_E^p x - J_E^p y, x \rangle. \end{aligned}$$

We know the following inequality which was proved by Xu [31].

Lemma 2.1. *[31] Let $x, y \in E$. If E is q-uniformly smooth, then there exists $C_q > 0$ such that*

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_E^q(x) \rangle + C_q\|y\|^q.$$

Let $x, y, z \in E$, one can easily get

$$\Delta_p(x, y) + \Delta_p(y, z) - \Delta_p(x, z) = \langle x - y, J_E^p z - J_E^p y \rangle, \tag{2.1}$$

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_E^p x - J_E^p y \rangle \tag{2.2}$$

and

$$\Delta_p(x, y) = \frac{\|x\|^p}{p} + \frac{\|y\|^p}{q} - \langle x, J_E^p(y) \rangle, \tag{2.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

For the p-uniformly convex space, the metric and Bregman distance has the following relation (see [24]):

$$\tau\|x - y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J_E^p x - J_E^p y \rangle, \tag{2.4}$$

where $\tau > 0$ is some fixed number.

Proposition 2.2. *[5, 12] Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that $\Delta_p(x_n, y_n) \to 0$. If $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Let C be a nonempty, closed and convex subset of E. The metric projection

$$P_C x = \operatorname{argmin}_{y\in C}\|x - y\|, \quad x \in E,$$

is the unique minimizer of the norm distance, which can be characterized by a variational inequality:

$$\langle J_E^p(x - P_C x), z - P_C x\rangle \le 0, \quad \forall z \in C. \tag{2.5}$$

Likewise, one can define the Bregman projection:

$$\Pi_C x = \operatorname{argmin}_{y\in C}\Delta_p(x, y), \quad x \in E,$$

as the unique minimizer of the Bregman distance (see [25]). The Bregman projection can also be characterized by a variational inequality:

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x\rangle \le 0, \quad \forall z \in C. \tag{2.6}$$

Moreover, we have

$$\Delta_p(\Pi_C x, z) \le \Delta_p(x, z) - \Delta_p(x, \Pi_C x), \quad \forall z \in C. \tag{2.7}$$

Let E be a strictly convex, smooth and reflexive Banach space. Following [2, 9], we make use of the function $V_p : E^* \times E \to [0, +\infty)$, which is defined by

$$V_p(\overline{x}, x) = \tfrac{1}{q}\|\overline{x}\|^q - \langle \overline{x}, x\rangle + \tfrac{1}{p}\|x\|^p, \quad \forall x \in E, \ \ \overline{x} \in E^*,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then V_p is nonnegative and

$$V_p(\overline{x}, x) = \Delta_p(J_{E^*}^q(\overline{x}), x) \tag{2.8}$$

for all $x \in E$ and $\overline{x} \in E^*$. Moreover, using the subdifferential inequality for $f(x) = \frac{1}{q}\|x\|^q, \ \ x \in E^*$, we have

$$\langle J_E^q(x), y\rangle \le \frac{1}{q}\|x + y\|^q - \frac{1}{q}\|x\|^q, \quad \forall x, y \in E^*. \tag{2.9}$$

Using (2.9), we have

$$V_p(\overline{x}, x) + \langle \overline{y}, J_{E^*}^q(\overline{x}) - x\rangle \le V_p(\overline{x} + \overline{y}, x) \tag{2.10}$$

for all $x \in E$ and $\bar{x}, \bar{y} \in E^*$ (see, for example, [27, 29]). In addition, V_p is convex in the first variable since $\forall z \in E$,

$$\Delta_p\left(J_{E^*}^q\left(\sum_{i=1}^N t_i J_E^p(x_i)\right), z\right) = V_p\left(\sum_{i=1}^N t_i J_E^p(x_i), z\right) \le \sum_{i=1}^N t_i \Delta_p(x_i, z), \tag{2.11}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

Let C be a convex subset of int domf_p, where $f_p(x) = \frac{1}{p}\|x\|^p$, $2 \le p < \infty$ and let T be a salf-mapping of C. A point $p \in C$ is said to be an *asymptotic fixed point* (please, see [10, 22]) of T if C contains a sequence $\{x_n\}_{n=1}^\infty$ which converges weakly to p and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

Definition 2.3. *A nonlinear mapping T with a nonempty asymptotic fixed point set is said to be: (i) left Bregman strongly nonexpansive (L-BSNE) (see [16, 17]) with respect to a nonempty $\widehat{F}(T)$ if*

$$\Delta_p(Tx,\bar{x}) \le \Delta_p(x,\bar{x}),\ \forall x \in C,\ \bar{x} \in \widehat{F}(T)$$

and if whenever $\{x_n\} \subset C$ is bounded, $\bar{x} \in \widehat{F}(T)$ and

$$\lim_{n\to\infty}(\Delta_p(x_n,\bar{x}) - \Delta_p(Tx_n,\bar{x})) = 0,$$

it follows that

$$\lim_{n\to\infty}\Delta_p(x_n,Tx_n) = 0.$$

(ii) An operator $T : C \to E$ is said to be: left Bregman firmly nonexpansive (L-BFNE) if

$$\langle J_p^E(Tx) - J_p^E(Ty), Tx - Ty\rangle \ \le \langle J_p^E(Tx) - J_p^E(Ty), x - y\rangle$$

for any $x, y \in C$.

The class of left Bregman strongly nonexpansive mappings is of particular significance in fixed point, iteration and convex optimization theories mainly because it is closed under composition. For more information and examples of L-BSNE and L-BFNE operators. From [16, 17], we know that every left Bregman firmly nonexpansive mapping is left Bregman strongly nonexpansive if $F(T) = \widehat{F}(T)$.

We also need the following tools in analysis which will be used in the sequel.

Lemma 2.4. *[15] Let $\{s_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ which satisfies $s_{n_i} < s_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\ge n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \le n : s_k < s_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \le n_0 : s_k < s_{k+1}\} \ne \emptyset$. Then, the following hold:

(i) $\tau(n_0) \le \tau(n_0+1) \le \ldots$ and $\tau(n) \to \infty$;

(ii) $s_{\tau(n)} \le s_{\tau(n)+1}$ and $s_n \le s_{\tau(n)+1}, \forall n \ge n_0$.

Lemma 2.5. *[31] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation :*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n,\ n \ge 1,$$

where (i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty}\alpha_n = \infty$; (ii) $\limsup_{n\to\infty}\sigma_n \le 0$; (iii) $\gamma_n \ge 0$; $(n \ge 1)$, $\sum_{n=1}^{\infty}\gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

We shall adopt the following notations in this paper:

- $x_n \to x$ means that $x_n \to x$ strongly;
- $x_n \rightharpoonup x$ means that $x_n \to x$ weakly;
- $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ is the weak ω-limit set of the sequence $\{x_n\}_{n=1}^{\infty}$.

3 Main results

In this section, we prove strong convergence theorem for the split feasibility problem in Banach spaces.

Theorem 3.1. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Let T be a left Bregman strongly nonexpansive mapping of C into it self such that $F(T) = \widehat{F}(T)$ and $F(T) \cap \Omega \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in (0,1). For a fixed $u \in E_1$, let sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be iteratively generated by $u_1 \in E_1$,*

$$\begin{cases} x_n = \Pi_C J_{E_1^*}^q [J_{E_1}^p(u_n) - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \nabla f(u_n)], \\ u_{n+1} = \Pi_C J_{E_1^*}^q (\alpha_n J_{E_1}^p(u) + (1-\alpha_n) J_{E_1}^p(Tx_n)), \quad n \geq 1, \end{cases} \tag{3.1}$$

where $f(u_n) = \frac{1}{p}\|(I - P_Q)Au_n\|^p$, $\nabla f(u_n) = A^* J_{E_2}^p(Au_n - P_Q(Au_n))$. If $\alpha_n \to 0$. $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\{\rho_n\} \subset (0, \infty)$ satisfies

$$\inf_n \rho_n(pq - C_q \rho_n^{q-1}) > 0.$$

Then the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to an element $x^* \in F(T) \cap \Omega$, where $x^* = \Pi_{F(T)\cap\Omega} u$.

Proof. We note that $\nabla f(u_n) = A^* J_{E_2}^p(Au_n - P_Q(Au_n))$ for all $n \in \mathbb{N}$. Set

$$y_n = J_{E_1}^p(u_n) - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \nabla f(u_n)$$

for all $n \in \mathbb{N}$. We see that $(p-1)q = p$. Then, by Lemma 2.1, we have

$$\begin{aligned} ||y_n||^q &= ||J_{E_1}^p(u_n) - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \nabla f(u_n)||^q \\ &\leq ||u_n||^p - q\rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \langle u_n, \nabla f(u_n)\rangle + C_q \rho_n^q \frac{f^{(p-1)q}(u_n)}{||\nabla f(u_n)||^{pq}} ||\nabla f(u_n)||^q \\ &= ||u_n||^p - q\rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \langle u_n, \nabla f(u_n)\rangle + C_q \rho_n^q \frac{f^p(u_n)}{||\nabla f(u_n)||^p}. \end{aligned} \tag{3.2}$$

Set $v_n = J_{E_1^*}^q [J_{E_1}^p(u_n) - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p} \nabla f(u_n)]$ for all $n \geq 1$. Then, we have $x_n = \Pi_C v_n$ for all $n \geq 1$.

Let $x^* = \Pi_{F(T)\cap\Omega} u$. Then by (3.2), we have

$$\begin{aligned}
\Delta_p(x_n, x^*) &\leq \Delta_p(v_n, x^*) = \Delta_p(J^q_{E_1^*}[J^p_{E_1}(u_n) - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p}\nabla f(u_n)], x^*) \\
&= \frac{||x^*||^p}{p} + \frac{1}{q}||J^p_{E_1}(u_n) - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p}\nabla f(u_n)||^q - \langle J^p_{E_1}(u_n), x^*\rangle \\
&\quad + \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p}\langle x^*, \nabla f(u_n)\rangle \\
&\leq \frac{1}{q}||u_n||^p - \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p}\langle u_n, \nabla f(u_n)\rangle + \frac{C_q}{q}\rho_n^q \frac{f^p(u_n)}{||\nabla f(u_n)||^p} \\
&\quad - \langle x^*, J^p_{E_1}(u_n)\rangle + \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p}\langle x^*, \nabla f(u_n)\rangle + \frac{\|x^*\|^p}{p} \\
&= \frac{1}{q}||u_n||^p - \langle x^*, J^p_{E_1}(u_n)\rangle + \frac{\|x^*\|^p}{p} + \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p}\langle x^* - u_n, \nabla f(u_n)\rangle \\
&\quad + \frac{C_q}{q}\rho_n^q \frac{f^p(u_n)}{||\nabla f(u_n)||^p} \\
&= \Delta_p(u_n, x^*) + \rho_n \frac{f^{p-1}(u_n)}{||\nabla f(u_n)||^p}\langle x^* - u_n, \nabla f(u_n)\rangle + \frac{C_q}{q}\rho_n^q \frac{f^p(u_n)}{||\nabla f(u_n)||^p}. \qquad (3.3)
\end{aligned}$$

On the other hand, we see that

$$\begin{aligned}
\langle \nabla f(u_n), x^* - u_n\rangle &= \langle A^* J^p_{E_2}(Au_n - P_Q(Au_n)), x^* - u_n\rangle \\
&= \langle J^p_{E_2}(Au_n - P_Q(Au_n)), Ax^* - Au_n\rangle \\
&= \langle J^p_{E_2}(Au_n - P_Q(Au_n)), P_Q(Au_n) - Au_n\rangle \\
&\quad + \langle J^p_{E_2}(Au_n - P_Q(Au_n)), Ax^* - P_Q(Au_n)\rangle \\
&\leq -\|Au_n - P_Q(Au_n)\|^p = -pf(u_n). \qquad (3.4)
\end{aligned}$$

By (3.3) and (3.4), we obtain

$$\begin{aligned}
\Delta_p(x_n, x^*) \leq \Delta_p(v_n, x^*) &\leq \Delta_p(u_n, x^*) + \frac{C_q}{q}\rho_n^q \frac{f^p(u_n)}{\|\nabla f(u_n)\|^p} - \rho_n \frac{pf^p(u_n)}{\|\nabla f(u_n)\|^p} \\
&= \Delta_p(u_n, x^*) + \left(\frac{C_q}{q}\rho_n^q - \rho_n p\right)\frac{f^p(u_n)}{\|\nabla f(u_n)\|^p}. \qquad (3.5)
\end{aligned}$$

Since $\inf_n \rho_n(pq - C_q\rho_n^{q-1}) > 0$, we have

$$\Delta_p(x_n, x^*) \leq \Delta_p(u_n, x^*), \quad \forall n \geq 1.$$

Now using (3.1), we have

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) \leq \Delta_p(u_{n+1}, x^*) &\leq \Delta_p(J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n)), x^*) \\
&\leq \alpha_n \Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(Tx_n, x^*) \\
&\leq \alpha_n \Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(x_n, x^*) \qquad (3.6) \\
&\leq \max\{\Delta_p(u, x^*), \Delta_p(x_n, x^*)\} \\
&\vdots
\end{aligned}$$

$$\leq \max\{\Delta_p(u,x^*),\Delta_p(x_1,x^*)\}.$$

Hence $\{u_n\}_{n=1}^{\infty}$ is bounded. Also $\{x_n\}_{n=1}^{\infty}$ is bounded.

Let $b_n := J^q_{E_1^*}(\alpha_n J^p_{E_1}(u)+(1-\alpha_n)J^p_{E_1}(Tx_n)),\ \ n\geq 1$. Then we obtain

$$\begin{aligned}\Delta_p(b_n,Tx_n) &\leq \alpha_n\Delta_p(u,Tx_n)+(1-\alpha_n)\Delta_p(Tx_n,Tx_n)\\ &= \alpha_n\Delta_p(u,Tx_n)\to 0,\quad n\to\infty.\end{aligned}$$

Set $w_n=\alpha_n J^p_{E_1}(u)+(1-\alpha_n)J^p_{E_1}(Tx_n)$ for all $n\geq 1$. We next consider the following estimation:

$$\begin{aligned}\Delta_p(x_{n+1},x^*)\leq\Delta_p(u_{n+1},x^*) &= \Delta_p(\Pi_C b_n,x^*)\leq\Delta_p(b_n,x^*)-\Delta_p(b_n,\Pi_C b_n)\\ &= \Delta_p(J^q_{E_1^*}[\alpha_n J^p_{E_1}(u)+(1-\alpha_n)J^p_{E_1}(Tx_n)],x^*)-\Delta_p(b_n,\Pi_C b_n)\\ &= V_p(\alpha_n J^p_{E_1}(u)+(1-\alpha_n)J^p_{E_1}(Tx_n),x^*)-\Delta_p(b_n,\Pi_C b_n)\\ &\leq V_p(\alpha_n J^p_{E_1}(u)+(1-\alpha_n)J^p_{E_1}(Tx_n)-\alpha_n(J^p_{E_1}(u)-J^p_{E_1}(x^*)),x^*)\\ &\quad+\alpha_n\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),J^q_{E_1^*}(w_n)-x^*\rangle-\Delta_p(b_n,\Pi_C b_n)\\ &= V_p(\alpha_n J^p_{E_1}(x^*)+(1-\alpha_n)J^p_{E_1}(Tx_n),x^*)\\ &\quad+\alpha_n\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),J^q_{E_1^*}(w_n)-x^*\rangle-\Delta_p(b_n,\Pi_C b_n)\\ &\leq (1-\alpha_n)V_p(J^p_{E_1}(Tx_n),x^*)\\ &\quad+\alpha_n\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),J^q_{E_1^*}(w_n)-x^*\rangle-\Delta_p(b_n,\Pi_C b_n)\\ &= (1-\alpha_n)\Delta_p(Tx_n,x^*)+\alpha_n\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),J^q_{E_1^*}(w_n)-x^*\rangle\\ &\quad-\Delta_p(b_n,\Pi_C b_n)\\ &\leq (1-\alpha_n)\Delta_p(x_n,x^*)+\alpha_n\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),J^q_{E_1^*}(w_n)-x^*\rangle\\ &\quad-\Delta_p(b_n,\Pi_C b_n).\end{aligned}\tag{3.7}$$

Let $s_n=\Delta_p(x_n,x^*)\ \ \forall n\in\mathbb{N}$. Then, by (3.7), we have

$$s_{n+1}\leq(1-\alpha_n)s_n+\alpha_n\langle J^p_{E_1}(u)-J^p_{E_1}(x^*),J^q_{E_1^*}(w_n)-x^*\rangle-\Delta_p(b_n,\Pi_C b_n).\tag{3.8}$$

We next consider the following two cases:

Case 1: Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\Delta_p(x_n,x^*)\}_{n=n_0}^{\infty}$ is non-increasing. Then $\{\Delta_p(x_n,x^*)\}_{n=1}^{\infty}$ converges and $\Delta_p(x_n,x^*)-\Delta_p(x_{n+1},x^*)\to 0,\ n\to\infty$. Now, from (3.5), we obtain

$$(\rho_n p-\frac{C_q}{q}\rho_n^q)\frac{f^p(u_n)}{\|\nabla f(u_n)\|^p}\leq\Delta_p(u_n,x^*)-\Delta_p(x_n,x^*).\tag{3.9}$$

Also, from (3.6), we have

$$\Delta_p(u_{n+1},x^*)\leq\alpha_n\Delta_p(u,x^*)+\Delta_p(x_n,x^*).\tag{3.10}$$

Putting (3.10) into (3.9), we have

$$\begin{aligned}(\rho_n p-\frac{C_q}{q}\rho_n^q)\frac{f^p(u_n)}{\|\nabla f(u_n)\|^p} &\leq \Delta_p(u_n,x^*)-\Delta_p(x_n,x^*)\\ &\leq \alpha_{n-1}\Delta_p(u,x^*)+\Delta_p(x_{n-1},x^*)-\Delta_p(x_n,x^*).\end{aligned}\tag{3.11}$$

By $\inf_n \rho_n(pq-C_q\rho_n^{q-1})>0$ and (3.11), we have

$$0 < (\rho_n p - \frac{C_q}{q}\rho_n^q)\frac{f^p(u_n)}{\|\nabla f(u_n)\|^p} \leq \alpha_{n-1}\Delta_p(u,x^*) + \Delta_p(x_{n-1},x^*) - \Delta_p(x_n,x^*) \to 0,\ n\to\infty.$$

It follows that $f(u_n) \to 0,\ n\to\infty$, since $\{\|\nabla f(u_n)\|\}$ is bounded. Hence

$$\lim_{n\to\infty} \|Au_n - P_Q(Au_n)\| = 0. \tag{3.12}$$

From (3.8), we have

$$0 \leq \Delta_p(b_n, \Pi_C b_n) \leq (s_n - s_{n+1}) + \alpha_n[\langle J_{E_1}^p(u) - J_{E_1}^p(x^*), J_{E_1^*}^q(w_n) - x^*\rangle - s_n] \to 0,\ n\to\infty.$$

Hence, by Proposition 2.2, we obtain

$$\|b_n - \Pi_C b_n\| \to 0,\ n\to\infty. \tag{3.13}$$

It also follows that

$$\begin{aligned} 0 \leq \|J_{E_1}^p(v_n) - J_{E_1}^p(u_n)\| &= \|J_{E_1}^p(u_n) - \rho_n\frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\nabla f(u_n) - J_{E_1}^p(u_n)\| \\ &= \|\rho_n\frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\nabla f(u_n)\| \to 0,\ n\to\infty. \end{aligned}$$

Therefore, we obtain

$$\lim_{n\to\infty} \|J_{E_1}^p(v_n) - J_{E_1}^p(u_n)\| = 0.$$

Since $J_{E_1^*}^q$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} \|v_n - u_n\| = 0.$$

Furthermore, we have from (2.7), (3.5) and (3.6) that

$$\begin{aligned} \Delta_p(v_n,x_n) = \Delta_p(v_n,\Pi_C v_n) &\leq \Delta_p(v_n,x^*) - \Delta_p(x_n,x^*) \\ &\leq \Delta_p(u_n,x^*) - \Delta_p(x_n,x^*) \\ &\leq \alpha_{n-1}M^* + \Delta_p(x_{n-1},x^*) - \Delta_p(x_n,x^*) \to 0,\ n\to\infty, \end{aligned}$$

for some $M^* > 0$. By Proposition 2.2, we have that $\|v_n - x_n\| \to 0,\ n\to\infty$.
Hence,

$$\|x_n - u_n\| \leq \|x_n - v_n\| + \|v_n - u_n\| \to 0,\ n\to\infty.$$

Observe that $\Delta_p(x_{n+1},x^*) \leq \Delta_p(u_{n+1},x^*) \leq \alpha_n\Delta_p(u,x^*) + (1-\alpha_n)\Delta_p(Tx_n,x^*)$.
It then follows that

$$\begin{aligned} \Delta_p(x_n,x^*) - \Delta_p(Tx_n,x^*) &= \Delta_p(x_n,x^*) - \Delta_p(x_{n+1},x^*) + \Delta_p(x_{n+1},x^*) - \Delta_p(Tx_n,x^*) \\ &\leq \Delta_p(x_n,x^*) - \Delta_p(x_{n+1},x^*) + \alpha_n(\Delta_p(u,x^*) - \Delta_p(Tx_n,x^*)) \\ &\to 0,\ n\to\infty. \end{aligned}$$

Then we obtain

$$\lim_{n\to\infty} \Delta_p(x_n,Tx_n) = 0.$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to z. Now, since $x_{n_j} \rightharpoonup z$ and $\lim_{n\to\infty} \|x_n - u_n\| = 0$, we obtain a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ that $u_{n_j} \rightharpoonup z$. Since $F(T) = \widehat{F}(T)$, we have $z \in F(T)$.

Next, we show that $z \in \Omega$. From (2.2), (2.4) and (2.6), we have

$$\begin{aligned}
\Delta_p(z, \Pi_C z) &\leq \langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), z - \Pi_C z\rangle \\
&= \langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), z - u_{n_j}\rangle + \langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), u_{n_j} - \Pi_C u_{n_j}\rangle \\
&\quad + \langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), \Pi_C u_{n_j} - \Pi_C z\rangle \\
&\leq \langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), z - u_{n_j}\rangle + \langle J_{E_1}^p(z) - J_{E_1}^p(\Pi_C z), u_{n_j} - \Pi_C u_{n_j}\rangle \\
&\to 0,
\end{aligned}$$

as $j \to \infty$. So we have $\Delta_p(z, \Pi_C z) = 0$. Thus, $z \in C$. Let us now fix $x \in C$ such that $Ax \in Q$. Then

$$\begin{aligned}
\|Au_{n_j} - P_Q(Au_{n_j})\|^p &= \langle J_{E_2}^p(Au_{n_j} - P_Q(Au_{n_j})), Au_{n_j} - P_Q(Au_{n_j})\rangle \\
&= \langle J_{E_2}^p(Au_{n_j} - P_Q(Au_{n_j})), Au_{n_j} - Ax)\rangle \\
&\quad + \langle J_{E_2}^p(Au_{n_j} - P_Q(Au_{n_j}), Ax - P_Q(Au_{n_j})\rangle \\
&\leq \langle J_{E_2}^p(Au_{n_j} - P_Q(Au_{n_j})), Au_{n_j} - Ax)\rangle \\
&\leq M\|A^*(I - P_Q)Au_{n_j}\|^{p-1} \\
&\to 0,\ n \to \infty.
\end{aligned}$$

where $M > 0$ is sufficiently large number. It then follows from (2.5) that

$$\begin{aligned}
\|Az - P_Q(Az)\|^p &= \langle J_{E_2}^p(Az - P_Q(Az)), Az - P_Q(Az)\rangle \\
&= \langle J_{E_2}^p(Az - P_Q(Az)), Az - Au_{n_j}\rangle + \langle J_{E_2}^p(Az - P_Q(Az)), Au_{n_j} - P_Q(Au_{n_j})\rangle \\
&\quad + \langle J_{E_2}^p(Az - P_Q(Az), P_Q(Au_{n_j}) - P_Q(Az)\rangle \\
&\leq \langle J_{E_2}^p(Az - P_Q(Az)), Az - Au_{n_j}\rangle + \langle J_{E_2}^p(Az - P_Q(Az)), Au_{n_j} - P_Q(Au_{n_j})\rangle.
\end{aligned}$$

Since $u_{n_j} \rightharpoonup z$, $Au_{n_j} \rightharpoonup Az$ and $\|Au_{n_j} - P_Q(Au_{n_j})\| \to 0$, $j \to \infty$, it follows that

$$\|Az - P_Q(Az)\| = 0.$$

Hence, $Az \in Q$. This shows that $z \in \Omega$ and therefore $z \in F(T) \cap \Omega$.

Moreover, we see that

$$\Delta_p(x_n, b_n) \leq \alpha_n \Delta_p(x_n, u) + (1 - \alpha_n)\Delta_p(x_n, Tx_n) \to 0,\ \ n \to \infty.$$

It follows that $\|x_n - b_n\| \to 0$, $n \to \infty$. We next show that

$$\limsup_{n\to\infty} \langle J_{E_1}^p(u) - J_{E_1}^p(x^*), b_n - x^*\rangle \leq 0.$$

We choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle J_{E_1}^p(u) - J_{E_1}^p(x^*), x_n - x^*\rangle = \lim_{j\to\infty} \langle J_{E_1}^p(u) - J_{E_1}^p(x^*), x_{n_j} - x^*\rangle.$$

From $\|x_n - b_n\| \to 0, n \to \infty$ and (2.6), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), b_n - x^*\rangle &= \limsup_{n\to\infty}\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), x_n - x^*\rangle \\ &= \langle J^p_{E_1}(u) - J^p_{E_1}(x^*), z - x^*\rangle \leq 0. \qquad (3.14)\end{aligned}$$

Note that $\|J^p_{E_1}(Tx_n) - w_n\| = \alpha_n\|J^p_{E_1}(Tx_n) - J^p_{E_1}(u)\| \to 0,\ n \to \infty$. On the other hand, we see that

$$\begin{aligned}\|J^p_{E_1}(b_n) - w_n\| &= \|\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}(Tx_n) - w_n\| \\ &\leq \alpha_n\|J^p_{E_1}(u) - w_n\| + \|J^p_{E_1}(Tx_n) - w_n\| \to 0,\ n \to \infty.\end{aligned}$$

This shows that $\|b_n - J^q_{E_1^*}(w_n)\| \to 0,\ n \to \infty$. So we obtain by (3.14)

$$\limsup_{n\to\infty}\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), J^q_{E_1^*}(w_n) - x^*\rangle \leq 0. \qquad (3.15)$$

Now, using (3.8), (3.15) and Lemma 2.5, we obtain $\Delta_p(x_n, x^*) \to 0,\ n \to \infty$. Hence, $x_n \to x^*$ as $n \to \infty$. Also we have $\|u_n - x^*\| \leq \|u_n - x_n\| + \|x_n - x^*\| \to 0,\ n \to \infty$. Thus $u_n \to x^*\ as\ n \to \infty$.

Case 2: Assume that $\{s_n\}$ is not monotonically decreasing sequence, and let $\tau : \mathbb{N} \to \mathbb{N}$ be as in Lemma 2.4. We see that, by Lemma 2.4 *(ii)*

$$\begin{aligned}\Delta_p(x_{\tau(n)}, x^*) - \Delta_p(Tx_{\tau(n)}, x^*) &= \Delta_p(x_{\tau(n)}, x^*) - \Delta_p(x_{\tau(n)+1}, x^*) + \Delta_p(x_{\tau(n)+1}, x^*) \\ &\quad -\Delta_p(Tx_{\tau(n)}, x^*) \\ &\leq \alpha_n(\Delta_p(u, x^*) - \Delta_p(Tx_{\tau(n)}, x^*)) \\ &\to 0,\ n \to \infty.\end{aligned}$$

It then follows that

$$\lim_{n\to\infty}\Delta_p(x_{\tau(n)}, Tx_{\tau(n)}) = 0.$$

Similar to Case 1, we can show that $\|Au_{\tau(n)} - P_Q Au_{\tau(n)}\| \to 0,\ n \to \infty$ and

$$\limsup_{n\to\infty}\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), J^q_{E_1^*}(w_{\tau(n)}) - x^*\rangle \leq 0.$$

Also from (3.8), we have that

$$s_{\tau(n)+1} \leq (1 - \alpha_{\tau(n)})s_{\tau(n)} + \alpha_{\tau(n)}\langle J^p_{E_1}(u) - J^p_{E_1}(x^*), J^q_{E_1^*}(w_{\tau(n)}) - x^*\rangle,$$

which gives

$$s_{\tau(n)} \leq \langle J^p_{E_1}(u) - J^p_{E_1}(x^*), J^q_{E_1^*}(w_{\tau(n)}) - x^*\rangle.$$

So by Lemma 2.5, we obtain

$$\lim_{n\to\infty} s_{\tau(n)} = 0.$$

We next show that $\lim_{n\to\infty} s_{\tau(n)+1} = 0$. To show this, it suffices to prove that

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0, n \to \infty.$$

Indeed, by (3.13), we observe that

$$\|x_{\tau(n)} - u_{\tau(n)+1}\| \leqslant \|x_{\tau(n)} - b_{\tau(n)}\| + \|b_{\tau(n)} - \Pi_C b_{\tau(n)}\| + \|\Pi_C b_{\tau(n)} - u_{\tau(n)+1}\| \to 0,\ n \to \infty.$$

This shows that

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \leq \|x_{\tau(n)+1} - u_{\tau(n)+1}\| + \|u_{\tau(n)+1} - x_{\tau(n)}\| \to 0,\ n \to \infty.$$

From (2.1), it follows that

$$\Delta_p(x^*, x_{\tau(n)+1}) + \Delta_p(x_{\tau(n)+1}, x_{\tau(n)}) - \Delta_p(x^*, x_{\tau(n)}) = \langle x^* - x_{\tau(n)+1}, J^p_{E_1}(x_{\tau(n)}) - J^p_{E_1}(x_{\tau(n)+1})\rangle.$$

Hence

$$s_{\tau(n)+1} = \Delta_p(x^*, x_{\tau(n)+1}) \leq \Delta_p(x^*, x_{\tau(n)}) + \langle x^* - x_{\tau(n)+1}, J^p_{E_1}(x_{\tau(n)}) - J^p_{E_1}(x_{\tau(n)+1})\rangle \to 0.$$

Thus, by Lemma 2.4, we obtain $0 \leq s_n \leq s_{\tau(n)+1}$, which implies that $\lim_{n\to\infty} s_n = 0$. This shows that $x_n \to x^*$ as $n \to \infty$, and hence $u_n \to x^*$ as $n \to \infty$. We thus complete the proof. □

4 Numerical Experiments

In this section, we provide some numerical examples and illustrate its performance by using Algorithm (3.1). Firstly, numerical results are shown in different choices of the step-size ρ_n with different values u and u_1.

Example 4.1 Let $E_1 = E_2 = L_2([0, 1])$ with the inner product given by

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

Let

$$C := \{x \in L_2([0, 1]) : \|x\|_{L_2} \leq 1\}.$$

Then

$$\Pi_C(x) = P_C(x) = \begin{cases} x, & \|x\| \leq 1 \\ \frac{x}{\|x\|}, & \|x\| > 1. \end{cases}$$

Also, let

$$Q := \{x \in L_2([0, 1]) : \langle x, a\rangle = b\},$$

where $a = \frac{t}{2}$, $\quad b = 0$. Then

$$P_Q(x) = \frac{b - \langle a, x\rangle}{\|a\|_2^2} a + x.$$

Let us assume that $A : L_2([0, 1]) \to L_2([0, 1])$, $\ (Ax)(t) = \frac{x(t)}{2}$. Then A is a bounded linear operator and $A^* = A$. Suppose that we take operator T in Theorem 3.1 as $T := P_C$, the metric projection on C (please see [16, 17]). Take $\alpha_n = \frac{1}{n+1}$, $\forall n \geq 1$, then our iterative scheme (3.1) becomes

$$x_n = P_C[u_n - \rho_n \frac{f(u_n)}{\|\nabla f(u_n)\|^2} A^*(Au_n - P_Q(Au_n))]$$
$$u_{n+1} = P_C[\frac{u}{n+1} + (1 - \frac{1}{n+1})(P_C x_n)], \quad n \geq 1, \tag{4.1}$$

where $f(u_n) = \frac{1}{2}\|Au_n - P_Q(Au_n)\|^2$ and $\nabla f(u_n) = A^*(Au_n - P_Q(Au_n))$ for all $n \in \mathbb{N}$.

We now study the effect (in terms of convergence, number of iterations required and the cpu time) of the sequence $\{\rho_n\} \subset (0, \infty)$ on the iterative scheme by choosing different ρ_n such that $\inf_n \rho_n(4 - \rho_n) > 0$ in the following cases.

Case 1: $\rho_n = \frac{0.5n}{n+1}$;
Case 2: $\rho_n = \frac{n}{n+1}$;
Case 3: $\rho_n = \frac{2n}{n+1}$;
Case 4: $\rho_n = \frac{3.5n}{n+1}$.

The stopping criterion is defined by $E_n = \frac{1}{2}\|Au_n - P_Q(Au_n)\|^2_{L_2} < 10^{-3}$, or using stopping criterion $n = 1,000$. We choose different choices of u and u_1 as
Choice 1: $u = t$ and $u_1 = \sin(t) + t^2$;
Choice 2: $u = t^2$ and $u_1 = e^t + 2t$.

The numerical experiments, using our Algorithm (3.1), for each case and choice are reported in the following Table 1.

Table 1: Algorithm (3.1) with different cases of ρ_n and different choices of u and u_1

		Case 1	Case 2	Case 3	Case 4
Choice 1	No. of Iter.	26	14	7	4
	cpu (Time)	1.247811	0.647647	0.327002	0.191387
Choice 2	No. of Iter.	20	10	5	3
	cpu (Time)	0.950551	0.467636	0.235971	0.143973

The error plotting of E_n for each choice of u and u_1 is shown in Figure 1-2, respectively.

Figure 1: Different Cases with Choice 1

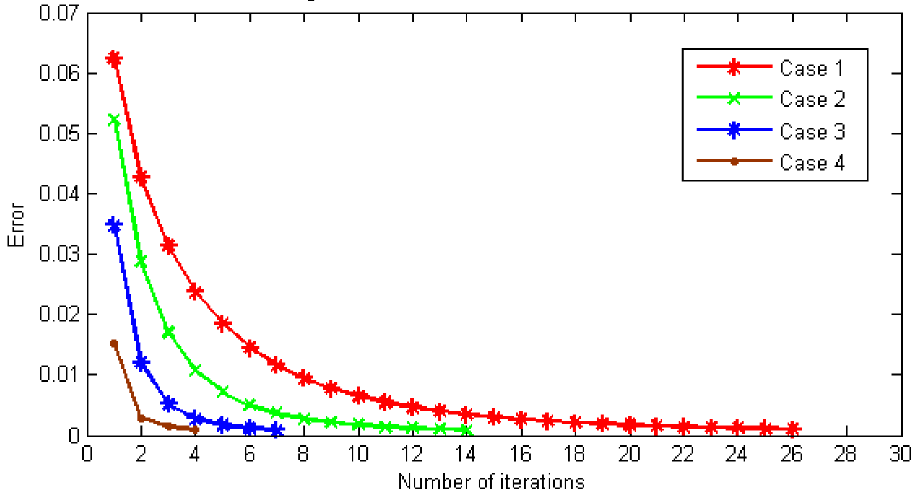
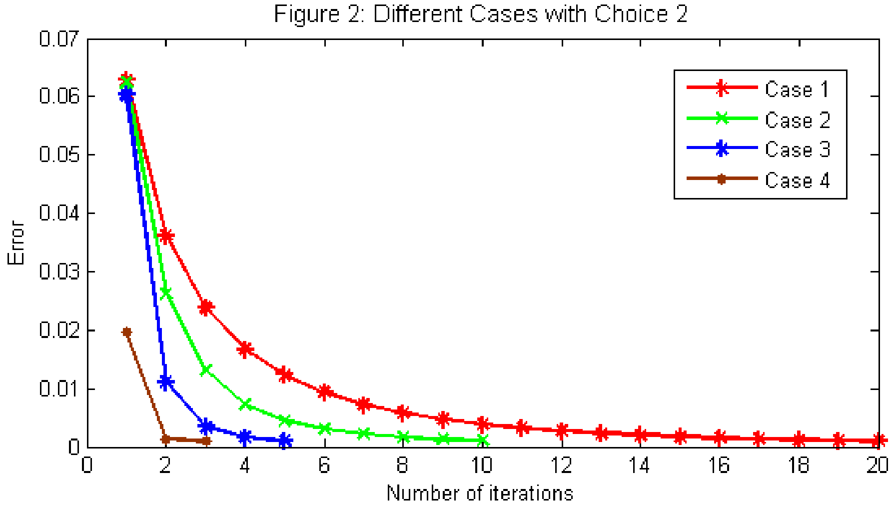


Figure 2: Different Cases with Choice 2



Remark 4.1. *From our numerical experiments, it is observed that the different choices of u and u_1 has no effect in terms of cpu run time for the convergence of our algorithm. It is observed that the number of iterations and the cpu run time are significantly decreasing starting from Case 1 to Case 4.*

Finally, we comparison of convergence of Algorithm (3.1) and Algorithm (1.6). Let $\alpha_n = \frac{1}{n+1}$, for algorithm (3.1), we take $\rho_n = \frac{0.5n}{n+1}$ and for algorithm (1.6), we take $t_n = 0.001$. We use stopping criterion $n = 1,000$. For points u and u_1 randomly, we obtain the following numerical results.

Table 2: Comparison of Algorithm (3.1) and Algorithm (1.6) in Example 4.1

			Algorithm (3.1)	Algorithm (1.6)
Choice 1	$u = t$	No. of Iter.	26	$>$ 1,000
	$u_1 = \sin(t) + t^2$	cpu (Time)	1.247811	-
Choice 2	$u = t^2$	No. of Iter.	20	$>$ 1,000
	$u_1 = e^t + 2t$	cpu (Time)	0.950551	-

The error plotting $n = 1,000$ of Algorithm (3.1) and Algorithm (1.6) for each choice is shown in Figure 3-4, respectively.

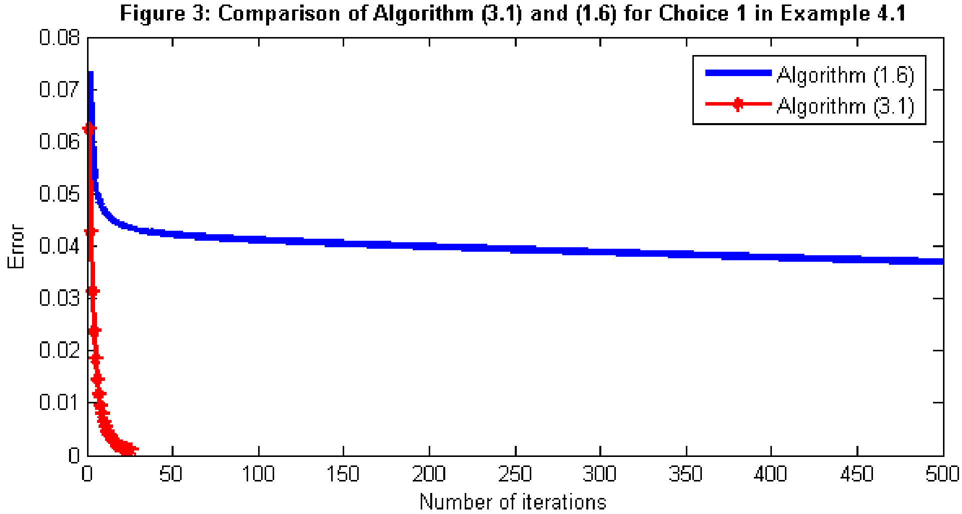
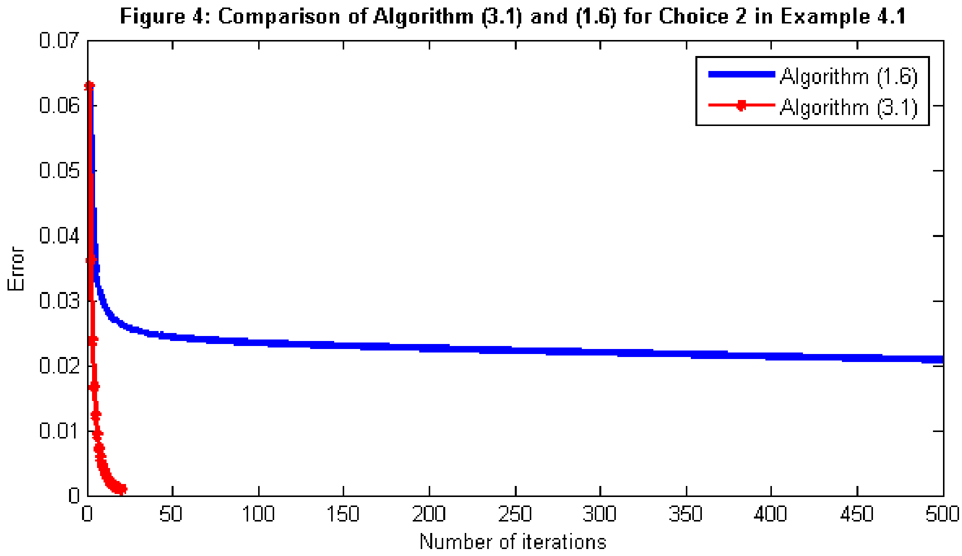


Figure 3: Comparison of Algorithm (3.1) and (1.6) for Choice 1 in Example 4.1



Remark 4.2. *In numerical experiment, it is revealed that the sequence generated by our proposed Algorithm (3.1) using the self-adaptive technique converges more quickly than by Algorithm (1.6) of Shehu et al. [26] does.*

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A self-adaptive method for solving the split feasibility problem and the fixed point problem of Bregman strongly nonexpansive mappings 2018