A STUDY ON THE PROOF OF CONVERGENCE THEOREM FOR SOLVING MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEM VIA A RESEARCH OF YONGHONG YAO, YEONG-CHENG LIOU AND YUH-JENN WU

NUTCHA APINYANANAN

PATCHARAPORN CHAIKHOD

An Independent Study Submitted in Partial Fulfillment

of the Requirements for the Bachelor

of Science Degree in Mathematics

December 2018

Advisor and Dean of School of Science have considered the independent study entitled "A study on The Proof of convergence Theorem for solving Mixed Equilibrium Problems and Fixed Point Problems" submitted in partial fulfillment of the requirements for Bachelor of Science Degree in Mathematics is hereby approved.

....

(Assoc. Prof. Dr. Prasit Cholamjiak)

Chairman

....

(Dr. Uamporn Witthayarat)

Committee and Advisor

....

(Asst. Prof. Dr. Watcharaporn Cholamjiak)

Committee

....

(Assoc. Prof. Preeyanan Sanpote)

Dean of School of Science

December 2018

ACKNOWLEDGEMENT

First of all, I would like to express my sincere appreciation to my supervisor, Dr. Uamporn Witthayarat , for her primary idea, guidance and motivation which enable me to carry out my study successfully.

I also thank to all of my teachers for their previous valuable lectures that give me more knowledge during my study at the Department of Mathematics, School of Science, University of Phayao.

I am thankful for all my friends with their help and warm friendship. Finally my graduation would not be achieved without best wish from my parents, who help me for everything and always gives me greatest love, willpower and financial support until this study completion

Nutcha apinyananan

Patcharaporn Chaikhod

ชื่อเรื่อง การศึกษาวิธีการพิสูจน์ทฤษฎีบทการลู่เข้าสำหรับการแก้ปัญหาสมดุลแบบผสมและปัญหาจุดตรึงผ่านงานวิจัยของ Yonghong Yao และ Yeong-Cheng Liou และ Yuh-Jenn

ผู้ศึกษาค้นคว้า นางสาวณัฐชา อภิญญาณานันท์
นางสาวพัชราภรณ์ ไชยโคตร

อาจารย์ที่ปรึกษา ดร.เอื้อมพร วิทยารัฐ

วิทยาศาสตรบัณฑิต สาขาคณิตศาสตร์

คำสำคัญ การลู่เข้าแบบเข้ม, การลู้เข้าแบบอ่อน, ปัญหาจุดตรึง

บทคัดย่อ

ในการศึกษาอิสระนี้ เราได้ศึกษาวิธีการประมาณค่าองค์ประกอบทั่วไปของเซตในจุดตรึงของ demicontractive mapping และเซตของการแก้ปัญหาของปัญหาสมดุลแบบผสม ขั้นแรกเราจะศึกษาวิธีการ Extragradient ที่นำมาใช้สำหรับการแก้ปัญหาสมดุลแบบผสมและปัญหาจุดตรึง ผ่านงานวิจัยของ Yonghong Yao และ Yeong-Cheng Liou และ Yuh-Jenn ต่อมาเราได้ทำการขยายขั้นตอนการพิสูจน์การลู่เข้าแบบเข้มของการศึกษาขั้นตอนภายใต้สมมติฐานบางส่วน เพื่อให้ง่ายต่อการทำความเข้าใจ สำหรับผู้ที่สนใจจะศึกษางานวิจัยนี้

Title A study on the proof of convergence theorem for solving mixed equilibrium problems and fixed point problem via a research fo Yonghong Yao, Yeong-Cheng Liou and Yuh-Jenn Wu

Author Miss Nutcha Apinyananan

Miss Patcharaporn Chaikhod

Advisor Dr.Uamporn Witthayarat

Bachelor of Science Program in Mathematics

Keywords Strong converges, Weak converges, Fixed point problems

ABSTRACT

In this independent study, we have studied the method to approximate a common element of the set of fixed point of a demicontractive mapping and the set of solutions of a mixed equilibrium problem. First, we studied an extragradient method for solving the mixed equilibrium problems and the fixed point problems via a research of Yonghong Yao, Yeong-Cheng Liou and Yuh-Jenn Wu. Subsequently, we extended the proof of its strong convergence of the proposed algorithm under some mild assumptions to make an easier understanding for those who are interested in this research.

List of contents

CHAPTER I

Introduction

1. Introduction

Let H be a real Hilbert space, C be a nonempty closed convex subset of H and let $\varphi : C \longrightarrow R$ be a real-valued function, $\ominus : C \times C \longrightarrow R$ be an equilibrium bifunction, that is, $\ominus(u,u) = 0$ for each $u \in C$.

We consider the following mixed equilibrium problem (MEP) which is to find

$x^* \in C$ such that

$$\ominus(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \ \forall y \in C. \tag{MEP}$$

If $\varphi \equiv 0$, this problem reduces to an equilibrium problem (EP), which is to find

$x^* \in C$ such that

$$\ominus(x^*, y) \geq 0, \ \forall y \in C. \tag{EP}$$

So that the set of solutions of (MEP) denoted by Ω and the set of solutions of (EP) denoted by Γ. The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and the equilibrium problems as special cases; see, for example, [1–5]. Some methods have been proposed to solve the equilibrium problems, see, for example, [5–21].

In 2005, Combettes and Hirstoaga [6] introduced an iterative algorithm of finding the best approximation to the initial data when $\Gamma \neq \emptyset$ and proved the strong convergence theorem.

S. Takahashi and W. Takahashi [8] introduced another iterative algorithm for finding a common element of the set of solutions of (EP) and the set of fixed points of a nonexpansive mapping in a real Hilbert space called the viscosity approximation method.

Let arbitrary initial $x_1 \in H$, define the sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\ominus (u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \tag{TT}$$

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, \qquad \forall n \geq 0.$$

Subsequently, they proved that the sequences $\{x_n\}$ and $\{u_n\}$ defined by (TT) converge strongly to $z \in Fix(S) \cap \Gamma$ with the following restrictions on algorithm parameters $\{\alpha_n\}$ and $\{r_n\}$ as follow:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\liminf_{n\to\infty} r_n > 0$;

(iii)$\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$; and $\sum_{n=0}^{\infty}|r_{n+1} - r_n| < \infty$.

Next, Zeng and Yao [16] introduced a new hybrid iterative algorithm for solving mixed equilibrium problems and fixed point problems and Mainge and Moudafi [22] introduced an iterative algorithm for solving equilibrium problems and fixed point problems.

On the other hand, Moudafi [23] showed the new method for solving the equilibrium problem (EP) and proved a weak convergence theorem.

Ceng et al. [24] introduced another iterative algorithm for finding an element of $Fix(S) \cap \Gamma$.

Let $S: C \to C$ be a k–strict pseudocontractive mapping for some $0 \leq k < 1$ such that $Fix(S) \cap \Gamma \neq \emptyset$. For given $x_1 \in H$, let the sequences $\{x_n\}$ and $\{u_n\}$ be generated iteratively by

$$\ominus (u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \tag{CAY}$$

$$x_{n+1} = \alpha_n u_n + (1-\alpha_n)Su_n, \ \forall n \geq 1,$$

where the parameters $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$;

(ii) $\{r_n\} \subset (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (CAY) converge weakly to an element of $Fix(S) \cap \Gamma$.

In this independent study, we are interested in the result of Yonghong Yao, Yeong-Cheng Liou and Yuh-Jenn Wu. So,we shall stydy the proof line of their strong convergence theorem and then we expand the proof line for an easier understanding of the theorem.

CHAPTER II

Preliminaries

2. Preliminaries

In case of extending the proof process of a strong convergence theorem, we shall introduce the necessariy definitions, lemmas and tools as follow.

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H.

Let $T: C \to C$ be a mapping. We use $Fix\,(T)$ to denote the set of the fixed points of T. Recall what following.

(i) T is called demicontractive if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - Tx\|^2 \tag{2.1}$$

for all $x \in C$ and $x^* \in Fix(T)$, which is equivalent to

$$\langle x - Tx, x - x^* \rangle \geq \frac{1-k}{2}\|x - Tx\|^2. \tag{2.2}$$

For such case, we also say that T is a k–demicontractive mapping

(ii) T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{2.3}$$

for all $x, y \in C$.

(iii) T is called quasi–nonexpansive if

$$\|Tx - x^*\| \leq \|x - x^*\| \tag{2.4}$$

for all $x \in C$ and $x^* \in Fix(T)$.

(iv) T is called strictly pseudocontractive if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2 \tag{2.5}$$

for all $x, y \in C$.

It is noting that the class of demicontractive mappings includes the class of the nonexpansive mappings, the quasi–nonexpansive mappings and the strictly pseudo–contractive mappings as special cases.

Let us also recall that T is called demiclosed if for any sequence $\{x_n\} \subset H$ and $x \in H$, we have

$$x_n \rightharpoonup x, \quad (I - T)x_n \to 0 \text{ strongly} \Longrightarrow x \in Fix(T). \tag{2.6}$$

It is well-known that the nonexpansive mappings, strictly pseudo-contractive mappings are all demiclosed. See, for example, [25-27].

An operator $A: C \to H$ is said to be δ-strongly monotone if there exists a positive constant δ such that

$$\langle Ax - Ay, x - y \rangle \geq \delta \|x - y\|^2 \tag{2.7}$$

for all $x, y \in C$.

Now, we concern the following problem: find $x^* \in Fix(T) \cap \Omega$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \forall x \in Fix(T) \cap \Omega. \tag{2.8}$$

In this paper, for solving problem (2.8) with an equilibrium bifunction $\ominus: C \times C \to R$, we assume that $\ominus$ satisfies the following conditions:

(H1) for each $x \in C, y \to \ominus\,(x, y)$ is conve;

(H2) $\ominus$ is monotone, that is, $\ominus\,(x, y) + \ominus\,(y, x) \leq 0$ for all $x, y \in C$;

(H3) for each fixed $y \in C, x \to \ominus\,(x, y)$ is concave and upper semicontinuous.

A mapping $\eta: C \times C \to H$ is called Lipschitz continuous, if there exists a constant $\lambda > 0$ such that

$$\|\eta(x, y)\| \leq \lambda \|x - y\|, \qquad \forall x, y \in C. \tag{2.9}$$

A differentiable function $K: C \to R$ on a convex set C is called

(i) η-convex if

$$K(y) - K(x) \geq \langle K'(x), \eta(y, x) \rangle, \forall x, y \in C, \tag{2.10}$$

where K' is the Frechet derivative of K at x;

(ii)η-strongly convex if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \langle K'(x), \eta(y, x) \rangle \geq \left(\frac{\sigma}{2}\right) \|x - y\|^2, \qquad \forall x, y \in C. \tag{2.11}$$

Futhermore, we need the following important and interesting tools for proving our main results.

Tool 1 More precisely, for a real vector space, an inner product $\langle\cdot,\ \cdot\rangle$ satisfies the following four properties. Let $u, v,$ and w be vectors and α be a scalar, then:

(i). $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$.

(ii). $\langle \alpha v, w\rangle = \alpha\langle v, w\rangle$.

(iii). $\langle v, w\rangle = \langle w, v\rangle$.

(iv). $\langle v, v\rangle \geq 0$ and equal if and only if $v = 0$.

Tool 2 (see,e.g.,Marino and Xu [10]). Let H be a real Hilbert space. There hold the following identities;

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2, \quad \forall x, y \in H.$

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - (1-t)\|x-y\|^2, \quad \forall x, y \in H.$

Tool 3 Given C is a closed convex subset of a Hilbert space H, a mapping $T: C \to H$ is firmly nonexpansive if for all $x, y \in C$

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle.$$

Tool 4 The limit inferior of a sequence $\{x_n\}$ is defined by

(i) $\lim_{n\to\infty} inf x_n \coloneqq \lim\limits_{n\to\infty} (\inf_{m\geq n} x_m)$

(ii) $\lim_{n\to\infty} inf x_n \coloneqq \sup_{n\geq 0}\ \inf_{m\geq n} x_m = sup\{inf\{x_m : m \geq n\} : n \geq 0\}$.

Silmilarly, the limit superior of $\{x_n\}$ is defined by

(i) $\lim_{n\to\infty} sup x_n \coloneqq \lim\limits_{n\to\infty} (\sup_{m\geq n} x_m)$

(ii) $\lim_{n\to\infty} sup x_n \coloneqq \inf_{n\geq 0}\ \sup_{m\geq n} x_m = inf\{sup\{x_m : m \geq n\} : n \geq 0\}$.

Alternatively, the notations

$$\lim_{n\to\infty} x_n \coloneqq \lim_{n\to\infty} inf x_n \text{ And } \lim_{n\to\infty} x_n \coloneqq \lim_{n\to\infty} sup x_n$$

are sometimes used.

Tool 5 A sequence of points $\{x_n\}$ in a Hilbert space H is said to converge weakly to a point x in H if

$$\langle x_n, y\rangle \to \langle x, y\rangle$$

for all y in H. Here,$\langle\cdot,\cdot\rangle$ is understood to be the inner product on the Hilbert space. The notation

$$x_n \rightharpoonup x.$$

is sometimes used to denote this kind of convergence.

Tool 6 A sequence x_n in a normed space X is said to be strongly convergent if there is an $x \in X$ such that

$$\lim_{n\to\infty}\|x_n - x\| = 0.$$

Tool 7 A sequence (a_n) is monotonic increasing if $a_{n+1} \geq a_n$ for all $n \in N$.

Tool 8 Bounded Sequences of Real Numbers

A sequences $\{a_n\}$ of numbers is said to be bounded above if there exists a real number $M \in \mathbb{R}$ such that $a_n \leq M$ for every $n \in \mathbb{N}$.

A sequences $\{a_n\}$ is said to be bounded below if there exists a real number $m \in \mathbb{R}$ such that $a_n \geq m$ for every $n \in \mathbb{N}$.

A sequences $\{a_n\}$ is said to be bounded if it is both bounded above and bounded below.

Tool 9 The following theorem is called Contraction Mapping Theorem or Banach Fixed Point Theorem.

Theorem 1. Consider a set $D \subset H$ and a function $: D \to H$. Assume

(i) D is closed (i.e., it contains all limit points of sequences in D).

(ii) $x \in D \;\Rightarrow\; g(x) \in D$.

(iii) The mapping g is a contraction on D: There exists $q < 1$such that

$$\forall x, y \in D:\ \|g(x) - g(y)\| \leq q\|x - y\|.$$

Let C be a nonempty closed convex subset of a real Hilbert space H, $\varphi: C \to R$ be real-valued function and $\ominus: C \times C \to R$ be an equilibrium bifunction. Let r be a positive number. For a given point $x \in C$, the problem for (MEP) is to find $y \in C$ such that

$$\ominus(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z,y)\rangle \geq 0, \forall z \in C. \quad (2.12)$$

Let $S_r : C \to C$ be the mapping such that for each $x \in C$, $S_r(x)$ is the solution set of the auxiliary problem, that is, $\forall x \in C$,

$$S_r(x) = \left\{ y \in C : \ominus(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z,y)\rangle \geq 0 \ \forall z \in C \right\}. \quad (2.13)$$

Lemma 2.1 ([16,28]). Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi: C \to R$ be a lower semicontinuous and convex functional. Let $\ominus: C \times C \to R$ be an equilibrium bifunction satisfying conditions (H1)–(H3). Assume what follows.

(i) $\eta: C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that

(a) $\eta(x,y) + \eta(y,x) = 0, \forall x, y \in C$,
(b) $\eta(\cdot,\cdot)$ is affine in the first variable,
(c) for each fixed $y \in C, x \longmapsto \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.

(ii) $K: C \to R$ is η–strongly convex with constant $\sigma > 0$ and its derivative K' is sequentially continuous from the weak topology to the strong topology.

(iii) For each $x \in C$, there exists a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \in C \backslash D_x$,

$$\ominus(y,z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x,y)\rangle < 0. \quad (2.14)$$

Then there hold the following:

(i) S_r is single–valued;

(ii) S_r is nonexpansive if K' is Lipschitz continuous with constant $v > 0$ such that $\sigma \geq \lambda v$ and

$$\langle K'(x_1) - K'(x_2), \eta(u_1, u_2)\rangle \geq \langle K'(u_1) - K'(u_2), \eta(u_1, u_2)\rangle, \forall (x_1, x_2) \in C \times C, \quad (2.15)$$

where $u_i = S_r(x_i)$ for $i = 1,2$;

(iii) $Fix(S_r) = \Omega$;

(v) Ω is closed and convex.

CHAPTER III

Main Results

3.1 Main Results

In this independent study, we focus on how to prove the strong convergence theorem. By studying the research of Yonghong Yao, Yeong-Cheng Liou and Yuh-Jenn Wu, we will extend proof lines in their theorem for an easier understanding for those who are interested in this research.

Let $A: H \to H$ be a mapping and $T: H \to H$ be a mapping. Let H be a real Hilbert space and C be a nonempty closed convex subset of H, $\varphi: H \to R$ be a lower semicontinuous and convex real-valued function, $\ominus: H \times H \to F$ be an equilibrium bifunction. In this section, we first introduce the following new iterative algorithm.

Algorithm 3.1 [25-27] Let r be a positive parameter. Let $\{\lambda_n\}$ be a sequence in $[0, \infty)$ and $\{\alpha_n\}$ be a sequence in $[0,1)$. Define the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ by the following manner:

$$x_0 \in C \text{ chosen arbitrarily,}$$

$$\ominus (z_n, x) + \varphi(x) - \varphi(z_n) + \frac{1}{r}\langle K'(z_n) - K'(x_n), \eta(x, z_n)\rangle \geq 0, \quad \forall n \in C,$$

$$y_n = z_n - \lambda_n A z_n \tag{3.1}$$

$$x_{n+1} = (1 - \alpha_n) y_n + \alpha_n T y_n.$$

Now we give a strong convergence result concerning Algorithm 3.1 as follows.

Theorem 3.2 [16,28] Let H be a real Hilbert space. Let $\varphi: H \to R$ be a lower semicontinuous and convex function. Let $\ominus: H \times H \to R$ be an equilibrium bifunction satisfying conditions (H1) - (H3). Let $A: H \to H$ be an L-Lipschitz continuous and δ-strongly monotone mapping and $T: H \to H$ be a demiclosed and k-demicontractive mapping such that $Fix(T) \cap \Omega \neq \emptyset$. Assume what follows.

(i) $\eta: H \times H \to H$ is Lipschitz continuous with constant

(a) $\eta(x,y) + \eta(y,x) = 0,\ \forall x,y \in H$

(b) $\eta(\cdot,\cdot)$ is affine in the first variable,

(c) for each fixed $y \in H,\ x \mapsto \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.

(ii) $K: H \to R$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $v > 0$ such that $\sigma \geq \lambda v$.

(iii) For each $x \in H$; there exist a bounded subset $D_x \subset H$ and $z_x \in H$ such that, for any $y \notin D_x$,

$$\ominus(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0 \tag{3.2}$$

(iv) $\alpha_n \in [\gamma, (1-k)/2]$ for some $\gamma > 0$, $lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated by (3.1) converge strongly to x^* which solves the problem (2.8) provided S_r is firmly nonexpansive.

Proof. First, we prove that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are all bounded. Without loss of generality, we may assume that $0 < \delta < L$. Given $\mu \in (0, 2\delta/L^2)$and $x,y \in H$ From **(Tool 2)**, we have

$$
\begin{aligned}
\|(\mu A - I)x - (\mu A - I)y\|^2 &= \|(\mu Ax - Ix) - (\mu Ay - Iy)\|^2 \\
&= \|(\mu Ax - x) - (\mu Ay - y)\|^2 \\
&= \|(\mu Ax - \mu Ay) - (x - y)\|^2 \\
&= \|(\mu Ax - \mu Ay)\|^2 - 2\langle \mu Ax - \mu Ay, x - y\rangle + \|x - y\|^2 \\
&= \mu^2\|Ax - Ay\|^2 + \|x - y\|^2 - 2\mu\langle Ax - Ay, x - y\rangle \\
&\leq \mu^2 L^2\|x - y\|^2 + \|x - y\|^2 - 2\mu\delta\|x - y\|^2 \qquad (3.3) \\
&= (1 - 2\mu\delta + \mu^2 L^2)\|x - y\|^2.
\end{aligned}
$$

That is,

$$\|(\mu A - I)x - (\mu A - I)y\| \leq \sqrt{1 - 2\mu\delta + \mu^2 L^2}\|x - y\|. \qquad (3.4)$$

Take $x^* \in Fix(T) \cap \Omega$. From (3.1), we have

$$\begin{aligned}
&\|y_{n+1} - (x^* - \lambda_{n+1}Ax^*)\| \\
&= \|(z_{n+1} - \lambda_{n+1}Az_{n+1}) - (x^* - \lambda_{n+1}Ax^*)\| \\
&= \|(z_{n+1} - x^*) - (\lambda_{n+1}Az_{n+1} - \lambda_{n+1}Ax^*)\| \\
&= \left\| z_{n+1} - x^* + \tfrac{\lambda_{n+1}}{\mu} z_{n+1} - \tfrac{\lambda_{n+1}}{\mu} z_{n+1} + \tfrac{\lambda_{n+1}}{\mu} x^* - \tfrac{\lambda_{n+1}}{\mu} x^* - \tfrac{\lambda_{n+1}}{\mu} \mu A z_{n+1} + \tfrac{\lambda_{n+1}}{\mu} \mu A x^* \right\| \\
&= \left\| \left(1 - \frac{\lambda_{n+1}}{\mu}\right)(z_{n+1} - x^*) - \frac{\lambda_{n+1}}{\mu}(\mu A z_{n+1} - z_{n+1} - \mu A x^* - x^*) \right\| \\
&= \left\| \left(1 - \frac{\lambda_{n+1}}{\mu}\right)(z_{n+1} - x^*) - \frac{\lambda_{n+1}}{\mu}((\mu A - I)z_{n+1} - (\mu A - I)x^*) \right\| \\
&\leq \left(1 - \frac{\lambda_{n+1}}{\mu}\right)\|z_{n+1} - x^*\| + \frac{\lambda_{n+1}}{\mu}\|(\mu A - I)z_{n+1} - (\mu A - I)x^*\| \\
&\leq \left(1 - \frac{\lambda_{n+1}}{\mu}\right)\|z_{n+1} - x^*\| + \frac{\lambda_{n+1}}{\mu}\sqrt{1 - 2\mu\delta + \mu^2 L^2}\|z_{n+1} - x^*\| \\
&= \left(1 - \tfrac{\lambda_{n+1}}{\mu} + \tfrac{\lambda_{n+1}}{\mu}\sqrt{1 - 2\mu\delta + \mu^2 L^2}\right)\|z_{n+1} - x^*\| \qquad (3.5) \\
&= \left(1 - \left(\tfrac{\lambda_{n+1}}{\mu}\right)\left(1 - \sqrt{1 - 2\mu\delta + \mu^2 L^2}\right)\right)\|z_{n+1} - x^*\|.
\end{aligned}$$

Therefore,

$$\|y_{n+1} - (x^* - \lambda_{n+1}Ax^*)\| \leq \left(1 - \left(\frac{\lambda_{n+1}\omega}{\mu}\right)\right)\|z_{n+1} - x^*\| \qquad (3.6)$$

where $\omega = \left(1 - \sqrt{1 - 2\mu\delta + \mu^2 L^2}\right) \in (0,1)$.

From **(Tool 2)** ,we have

$$\begin{aligned}
\|(z_{n+1} - x^*) - (x_{n+1} - x^*)\|^2 &= \|z_{n+1} - x^*\|^2 - 2\langle z_{n+1} - x^*, x_{n+1} - x^*\rangle + \|x_{n+1} - x^*\|^2 \\
\|z_{n+1} - x_{n+1}\|^2 &= \|z_{n+1} - x^*\|^2 - 2\langle z_{n+1} - x^*, x_{n+1} - x^*\rangle + \|x_{n+1} - x^*\|^2 \\
2\langle z_{n+1} - x^*, x_{n+1} - x^*\rangle &= \|z_{n+1} - x^*\|^2 + \|x_{n+1} - x^*\|^2 - \|z_{n+1} - x_{n+1}\|^2 \\
\langle z_{n+1} - x^*, x_{n+1} - x^*\rangle &= \tfrac{1}{2}(\|z_{n+1} - x^*\|^2 + \|x_{n+1} - x^*\|^2 - \|z_{n+1} - x_{n+1}\|^2. \qquad (3.7)
\end{aligned}$$

Note that $z_{n+1} = S_r x_{n+1}$ and S_r is firmly nonexpansive. Together with (3.7), we have

$$
\begin{aligned}
\|z_{n+1}-x^*\|^2 &= \|S_r x_{n+1} - S_r x^*\|^2 \\
&\le \langle x_{n+1}-x^*, S_r x_{n+1} - S_r x^*\rangle \\
&= \langle S_r x_{n+1} - S_r x^*, x_{n+1}-x^*\rangle \\
&= \langle z_{n+1}-x^*, x_{n+1}-x^*\rangle \\
&= \tfrac{1}{2}(\|z_{n+1}-x^*\|^2 + \|x_{n+1}-x^*\|^2 - \|x_{n+1}-z_{n+1}\|^2), \qquad (3.8)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|z_{n+1}-x^*\|^2 &\le \frac{1}{2}(\|z_{n+1}-x^*\|^2 + \|x_{n+1}-x^*\|^2 - \|x_{n+1}-z_{n+1}\|^2) \\
2\|z_{n+1}-x^*\|^2 - \|z_{n+1}-x^*\|^2 &\le \|x_{n+1}-x^*\|^2 - \|x_{n+1}-z_{n+1}\|^2 \\
\|z_{n+1}-x^*\|^2 &\le \|x_{n+1}-x^*\|^2 - \|x_{n+1}-z_{n+1}\|^2. \qquad (3.9)
\end{aligned}
$$

Since $\propto_n \in [\gamma, (1-k)/2]$ and $k \in [0,1)$ and $0 \le k < 1$, $0 \le \frac{k}{2} < \frac{1}{2}$,we have

$$
\begin{aligned}
\propto_n &\le \tfrac{1-k}{2} \\
\propto_n + k &\le \tfrac{1-k}{2} + k \\
&= \tfrac{1}{2} - \tfrac{k}{2} + k \\
&= \tfrac{1}{2} + \tfrac{k}{2} \\
&< \tfrac{1}{2} + \tfrac{1}{2} \\
&= 1. \qquad (3.10)
\end{aligned}
$$

From **(3.10)**, hence $\propto_n + k < 1$, such that

$$1 - (\propto_n + k) \ge 0. \qquad (3.11)$$

From **(2.2), (3.1)** and **(3.11)**, we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|(1-\propto_n)y_n + \propto_n Ty_n - x^*\|^2 \\
&= \|y_n - \propto_n y_n + \propto_n Ty_n - x^*\|^2 \\
&= \|(y_n - x^*) - (\propto_n y_n + \propto_n Ty_n)\|^2 \\
&= \|(y_n - x^*) - \propto_n (y_n + Ty_n)\|^2 \\
&= \|y_n - x^*\|^2 - 2\propto_n \langle y_n - x^*, y_n + Ty_n\rangle + \|\propto_n (y_n + Ty_n)\|^2 \\
&= \|y_n - x^*\|^2 - 2\propto_n \langle y_n + Ty_n, y_n - x^*\rangle + \propto_n^2 \|(y_n + Ty_n)\|^2
\end{aligned}
$$

$$\|x_{n+1} - x^*\|^2 \leq \|y_n - x^*\|^2 - 2 \propto_n \tfrac{1-k}{2} \|y_n + Ty_n\|^2 + {\propto_n}^2 \ \|(y_n + Ty_n)\|^2 \qquad (3.12)$$

$$= \|y_n - x^*\|^2 - \propto_n (1-k)\|y_n + Ty_n\|^2 + {\propto_n}^2 \ \|(y_n + Ty_n)\|^2$$

$$= \|y_n - x^*\|^2 - \propto_n (1 - k - \propto_n)\|y_n + Ty_n\|^2$$

$$\leq \|y_n - x^*\|^2.$$

From **(3.6) – (3.12)**, we have

$$\|y_{n+1} - x^*\| \leq \|y_{n+1} - x^* + \lambda_{n+1}Ax^* - \lambda_{n+1}Ax^*\|$$

$$\leq \|y_{n+1} - (x^* + \lambda_{n+1}Ax^*)\| + \|\lambda_{n+1}Ax^*\|$$

$$= \|y_{n+1} - (x^* + \lambda_{n+1}Ax^*)\| + \lambda_{n+1}\|Ax^*\|$$

$$\leq \left(1 - \tfrac{\lambda_{n+1}\omega}{\mu}\right)\|z_{n+1} - x^*\| + \lambda_{n+1}\|Ax^*\|$$

$$\leq \left(1 - \tfrac{\lambda_{n+1}\omega}{\mu}\right)\|x_{n+1} - x^*\| + \lambda_{n+1}\|Ax^*\|$$

$$\leq \left(1 - \tfrac{\lambda_{n+1}\omega}{\mu}\right)\|y_n - x^*\| + \lambda_{n+1}\|Ax^*\|$$

$$= \left(1 - \tfrac{\lambda_{n+1}\omega}{\mu}\right)\|y_n - x^*\| + \tfrac{\lambda_{n+1}\omega}{\mu}\left\{\tfrac{\mu}{\omega}\|Ax^*\|\right\} \qquad (3.13)$$

$$\leq max\left\{\|y_n - x^*\|, \frac{\mu}{\omega}\|Ax^*\|\right\}$$

$$\leq max\left\{\|y_0 - x^*\|, \tfrac{\mu}{\omega}\|Ax^*\|\right\}.$$

This implies that $\{y_n\}$ is bounded.

Therefore, from **(3.12)** and **(3.13)**,we have

$$\|x_{n+1}\| = \|x_{n+1} + x^* - x^*\|$$

$$= \|x_{n+1} - x^*\| + \|x^*\|$$

$$= \|y_{n+1} - x^*\| + \|x^*\|$$

$$= M + \|x^*\|, \qquad \forall n \in N. \qquad (3.14)$$

This implies that $\{x_n\}$ is bounded.

Therefore, from **(3.9)** and **(3.14),**we have

$$\|z_{n+1}\| = \|z_{n+1} + x^* - x^*\|$$

$$= \|z_{n+1} - x^*\| + \|x^*\|$$

$$\|z_{n+1}\| \le \|x_{n+1} - x^*\|^2 - \|x_{n+1} - z_{n+1}\|^2 + \|x^*\|$$
$$\le \|x_{n+1} - x^*\|^2 + \|x^*\|$$
$$\le M + \|x^*\|, \qquad \forall n \in N. \tag{3.15}$$

This implies that $\{z_n\}$ is bounded.

From (3.1),we can write $y_n - Ty_n = (1/\propto_n)(y_n - x_{n+1})$.Thus, from (3.12), we have

$$\|x_{n+1} - x^*\|^2 \le \|y_n - x^*\|^2 -\propto_n (1 - k -\propto_n)\|y_n - Ty_n\|^2$$
$$\le \|y_n - x^*\|^2 -\propto_n (1 - k -\propto_n) \left\|\frac{1}{\propto_n}(y_n - x_{n+1})\right\|^2$$
$$\le \|y_n - x^*\|^2 -\propto_n (1 - k -\propto_n)\frac{1}{{\propto_n}^2}\|(y_n - x_{n+1})\|^2$$
$$\le \|y_n - x^*\|^2 - \frac{1-k-\propto_n}{\propto_n}\|(y_n - x_{n+1})\|^2. \tag{3.16}$$

Since $\alpha_n \in (0, (1-k)/2]$, we have

$$0 < \propto_n \le \frac{1-k}{2}$$
$$\frac{1-k}{2} \ge \propto_n$$
$$1 - k \ge 2 \propto_n$$
$$1 - k \ge \propto_n + \propto_n$$
$$1 - k -\propto_n \ge \propto_n$$
$$\frac{1-k-\propto_n}{\propto_n} \ge 1. \tag{3.17}$$

Therefore $(1 - k -\propto_n)/\propto_n \ge 1$, from **(3.9)** and **(3.16)**, we obtain

$$\|x_{n+1} - x^*\|^2 \le \|y_n - x^*\|^2 - \left(\frac{1-k-\propto_n}{\propto_n}\right)\|y_n - x_{n+1}\|^2$$
$$\le \|y_n - x^*\|^2 - \|y_n - x_{n+1}\|^2$$
$$= \|z_n - \lambda_n Az_n - x^*\|^2 - \|z_n - \lambda_n Az_n - x_{n+1}\|^2$$
$$= \|z_n - x^* - \lambda_n Az_n\|^2 - \|z_n - x_{n+1} - \lambda_n Az_n\|^2$$
$$= \|z_n - x^*\|^2 - 2\lambda_n\langle Az_n, z_n - x^*\rangle + {\lambda_n}^2\|Az_n\|^2$$
$$-\|z_n - x_{n+1}\|^2 + 2\lambda_n\langle Az_n, z_n - x_{n+1}\rangle - {\lambda_n}^2\|Az_n\|^2$$
$$= \|z_n - x^*\|^2 - 2\lambda_n\langle x_{n+1} - x^*, Az_n\rangle - \|x_{n+1} - z_n\|^2$$

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 - 2\lambda_n\langle x_{n+1} - x^*, Az_n\rangle$$
$$-\|x_{n+1} - z_n\|^2. \tag{3.18}$$

We note that $\{x_n\}$ and $\{z_n\}$ are bounded. So there exists a constant $M \geq 0$ such that

$$|\langle x_{n+1} - x^*, Az_n\rangle| \leq M \quad \forall n \geq 0. \tag{3.19}$$

Consequently, we get

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 + \|x_{n+1} - z_n\|^2 + \|x_n - z_n\|^2 \leq 2M\lambda_n. \tag{3.20}$$

Now we divide into two cases to prove that $\{x_n\}$ converges strongly to x^*.

Case 1. Assume that the sequence $\{\|x_n - x^*\|\}$ is a monotone sequence. Then

$\{\|x_n - x^*\|\}$ is convergent. Setting $\lim_{n\to\infty}\|x_n - x^*\| = d$.

(i) If $d = 0$, then the desired conclusion is obtained.

(ii) Assume that $d > 0$. Clearly, we have

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 \to 0. \tag{3.21}$$

This together with $\lambda_n \to 0$ and (3.20), (3.21) implies that

$$\|x_{n+1} - z_n\|^2 + \|x_n - z_n\|^2 \to 0. \tag{3.22}$$

That is to say

$$\|x_{n+1} - z_n\| \to 0, \qquad \|x_n - z_n\| \to 0. \tag{3.23}$$

Let $z \in H$ be a weak limit point of $\{z_{n_k}\}$. Then there exists a subsequence of $\{z_{n_k}\}$, still denoted by $\{z_{n_k}\}$. Which weakly converges to z. Noting that $\lambda_n \to 0$, we also have

$$y_{n_k} = z_{n_k} - \lambda_{n_k}Az_{n_k} \rightharpoonup z. \tag{3.24}$$

Combining **(3.1)and (3.23)** we have

$$x_{n_k+1} = (1 - \alpha_n)y_{n_k} + \alpha_n Ty_{n_k}$$
$$\alpha_n Ty_{n_k} = x_{n_k+1} - (1 - \alpha_n)y_{n_k}$$
$$= x_{n_k+1} - \left(y_{n_k} - \alpha_n y_{n_k}\right)$$
$$= x_{n_k+1} - y_{n_k} + \alpha_n y_{n_k}$$

$$\alpha_n T y_{n_k} - \alpha_n y_{n_k} = x_{n_k+1} - y_{n_k}$$

$$\begin{aligned}
\|Ty_{n_k} - y_{n_k}\| &= \frac{1}{\alpha_{nk}}(x_{n+1} - y_n) \\
&= \frac{1}{\alpha_{nk}}\left(y_{n_k} - x_{n+1}\right) \\
&= \frac{1}{\alpha_{nk}}\left(x_{n_k+1} - y_{n_k}\right) \\
&= \frac{1}{\alpha_{nk}}\|x_{n_k+1} - z_{n_k} + \lambda_{n_k} A z_{n_k}\| \qquad (3.25) \\
&\leq \frac{1}{\alpha_{nk}}\|x_{n_k+1} - z_{n_k}\| + \frac{1}{\alpha_{nk}}\lambda_{n_k}\|Az_{n_k}\| \\
&\to 0.
\end{aligned}$$

Since T is demiclosed, then we obtain $z \in Fix(T)$.

Next we show that $z \in \Omega$ Since $z_n = S_r x_n$, we derive

$$\ominus(z_n, x) + \varphi(x) - \varphi(z_n) + \frac{1}{r}\langle K'(z_n) - K'(x_n), \eta(x, z_n)\rangle \geq 0, \qquad \forall x \in C. \qquad (3.26)$$

From the monotonicity of $\ominus$, we have

$$\frac{1}{r}\langle K'(z_n) - K'(x_n), \eta(x, z_n)\rangle + \varphi(x) - \varphi(z_n) \geq -\ominus(z_n, x) \geq \ominus(x, z_n) \qquad (3.27)$$

and hence

$$\frac{1}{r}\langle K'(z_n) - K'(x_n), \eta(x, z_n)\rangle + \varphi(x) - \varphi(z_n) \geq -\ominus(z_n, x) \geq \ominus(x, z_n). \qquad (3.28)$$

Since $\left(K'\left(z_{n_k}\right) - K'\left(x_{n_k}\right)\right)/r \to 0$ and $z_{n_k} \rightharpoonup z$, from the weak lower semicontinuity of φ and $\ominus(x, y)$ in the second variable y, we have

$$\ominus(x, z) + \varphi(z) - \varphi(x) \leq 0, \qquad \forall x \in C \qquad (3.29)$$

For $0 < t \leq 1$ and $x \in C$, let $x_t = tx + (1-t)z$. Since $x \in C$ and $z \in C$, we have $x_t \in C$ and hence $\ominus(x_t, z) + \varphi(z) - \varphi(x_t) \leq 0$ From the convexity of equilibrium bifunction $\ominus(x, y)$ in the second variable y, we have

$$\begin{aligned}
0 &= \ominus(x_t, x_t) + \varphi(x_t) - \varphi(x_t) \\
&= \ominus(x_t, tx + (1-t)z) + \varphi(tx + (1-t)z) - \varphi(x_t) \\
&\leq \ominus(x_t, tx) + \ominus(x_t, (1-t)z) + \varphi(tx) + \varphi((1-t)z) - \varphi(x_t) \\
&= t\ominus(x_t, x) + (1-t)\ominus(x_t, z) + t\varphi(x) + (1-t)\varphi(z) - \varphi(x_t)
\end{aligned}$$

$$0 \leq t\ominus(x_t, x) + (1-t)\left(\varphi(x_t) - \varphi(z)\right) + t\varphi(x) + (1-t)\varphi(z) - \varphi(x_t)$$

$$= t \ominus (x_t, x) + \varphi(x_t) - \varphi(z) - t\varphi(x_t) + t\varphi(z) + t\varphi(x) + \varphi(z) - t\varphi(z) - \varphi(x_t)$$

$$= t \ominus (x_t, x) - t\varphi(x_t) + t\varphi(x)$$

$$\leq t[\ominus (x_t, x) + \varphi(x) - \varphi(x_t)], \tag{3.30}$$

and hence $\ominus (x_t, x) + \varphi(x) - \varphi(x_t) \geq 0$. Then, we have

$$\ominus (z, x) + \varphi(x) - \varphi(z) \geq 0 \tag{3.31}$$

for all $x \in C$ and hence $z \in \Omega$.

Therefore, we have

$$z \in Fix(T) \cap \Omega. \tag{3.32}$$

Thus, if x^* is a solution of problem (2.8), we have

$$\lim_{k\to\infty} \inf\langle z_{n_k} - x^*, Ax^*\rangle = \langle z - x^*, Ax^*\rangle \geq 0. \tag{3.33}$$

Suppose that there exists another subsequence $\{z_{n_i}\}$ which weakly converges to z'. It is easily checked that $z' \in Fix(T) \cap \Omega$ and

$$\lim_{i\to\infty} \inf\langle z_{n_i} - x^*, Ax^*\rangle = \langle z' - x^*, Ax^*\rangle \geq 0. \tag{3.34}$$

Therefore, we have

$$\lim_{n\to\infty} \inf\langle z_n - x^*, Ax^*\rangle \geq 0. \tag{3.35}$$

Since A is δ-strongly monotone, we have

$$\begin{aligned}
\langle x_{n+1} - x^*, Az_n\rangle &= \langle x_{n+1} - x^* + z_n - z_n, Az_n + Ax^* - Ax^*\rangle \\
&= \langle x_{n+1} - z_n, Az_n + Ax^* - Ax^*\rangle + \langle z_n - x^*, Az_n + Ax^* - Ax^*\rangle \\
&= \langle z_n - x^*, Az_n - Ax^*\rangle + \langle z_n - x^*, Ax^*\rangle + \langle x_{n+1} - z_n, Az_n\rangle \\
&\geq \delta\|z_n - x^*\|^2 + \langle z_n - x^*, Ax^*\rangle + \langle x_{n+1} - z_n, Az_n\rangle. \qquad (3.36)
\end{aligned}$$

By **(3.9)** and **(3.23)** – **(3.36)**, we get

$$\langle x_{n+1} - x^*, Az_n\rangle \geq \delta\|z_n - x^*\|^2 + \langle z_n - x^*, Ax^*\rangle + \langle x_{n+1} - z_n, Az_n\rangle$$

$$\begin{aligned}
\langle x_{n+1} - x^*, Az_n\rangle &\geq \delta\|z_n - x^*\|^2 \\
&\geq \delta\|x_n - x^*\|^2 \\
&\geq \delta d^2
\end{aligned}$$

$$\lim_{n\to\infty} \inf\langle x_{n+1} - x^*, Az_n\rangle \geq \delta d^2. \tag{3.37}$$

Therefore,

$$\lim_{n\to\infty} \inf\langle x_{n+1} - x^*, Az_n\rangle \geq \delta d^2 - \epsilon$$

$$-2\lambda_n \lim_{n\to\infty} \inf\langle x_{n+1} - x^*, Az_n\rangle \leq -2\lambda_n(\delta d^2 - \epsilon),$$

(3.38)

From **(3.18), (3.22)** and **(3.38)**, for $0 < \varepsilon < \delta d^2$, we deduce that there exists a positive integer number n_0 large enough, when $n \geq n_0$,

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 - 2\lambda_n\langle x_{n+1} - x^*, Az_n\rangle - \|x_{n+1} - z_n\|^2 . \tag{3.39}$$

Then,

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 \leq -2\lambda_n\langle x_{n+1} - x^*, Az_n\rangle - \|x_n - z_n\|^2 - \|x_{n+1} - z_n\|^2$$

$$\leq -2\lambda_n(\delta d^2 - \varepsilon). \tag{3.40}$$

This implies that

$$\|x_{n+1} - x^*\|^2 - \left\|x_{n_0} - x^*\right\|^2 \leq -2(\delta d^2 - \epsilon) \sum_{k=n_0}^{n} \lambda_k . \tag{3.41}$$

Since $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\{x_n\}$ is bounded, hence the last inequality is a contraction. Therefore, $d = 0$, that is to say, $x_n \to x^*$.

Case 2. Assume that $\{\|x_n - x^*\|\}$ is not a monotone sequence. Set $\Gamma_n = \|x_n - x^*\|^2$ and let $\tau: N \to N$ be a mapping for all $n \geq n_0$ by

$$\tau(n) = max\{k \in N: k \leq n, \Gamma_k \leq \Gamma_{k+1}\}. \tag{3.42}$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$. From **(3.42)**, we have

$$\left\|x_{\tau(n)+1} - z_{\tau(n)}\right\|^2 + \left\|x_{\tau(n)} - z_{\tau(n)}\right\|^2 \leq 2M\lambda_{\tau(n)} \to 0, \tag{3.43}$$

Thus

$$\left\|x_{\tau(n)+1} - z_{\tau(n)}\right\| \to 0, \qquad \left\|x_{\tau(n)} - z_{\tau(n)}\right\| \to 0. \tag{3.44}$$

Therefore,

$$\left\|x_{\tau(n)+1} - z_{\tau(n)}\right\|^2 + \left\|x_{\tau(n)} - z_{\tau(n)}\right\|^2 \to 0$$

$$\left\|(x_{\tau(n)+1} - z_{\tau(n)}) - (x_{\tau(n)} - z_{\tau(n)})\right\|^2 \to 0$$

$$\left\|x_{\tau(n)+1} - z_{\tau(n)} - x_{\tau(n)} + z_{\tau(n)}\right\|^2 \to 0$$

$$\left\|x_{\tau(n)+1} - x_{\tau(n)}\right\| \to 0. \qquad (3.45)$$

From **(3.44)**,we have

$$\left\|x_{\tau(n)+1} - x_{\tau(n)} + x^* - x^*\right\| \to 0$$

$$\left\|x_{\tau(n)+1} - x^*\right\| + \left\|x_{\tau(n)} - x^*\right\| \to 0. \qquad (3.46)$$

Since $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, for all $n \geq n_0$, from **(3.18)**, we get

$$0 \leq \left\|x_{\tau(n)} - x^*\right\|^2 - \left\|x_{\tau(n)} - z_{\tau(n)}\right\|^2 - 2\lambda_{\tau(n)}\langle x_{\tau(n)+1} - x^*, Az_{\tau(n)}\rangle$$
$$-\left\|x_{\tau(n)+1} - z_{\tau(n)}\right\|^2 - \left\|x_{\tau(n)+1} - x^*\right\|^2$$

$$0 \geq \left\|x_{\tau(n)+1} - x^*\right\|^2 - \left\|x_{\tau(n)} - x^*\right\|^2 + \left\|x_{\tau(n)} - z_{\tau(n)}\right\|^2 + \left\|x_{\tau(n)+1} - z_{\tau(n)}\right\|^2$$
$$+2\lambda_{\tau(n)}\langle x_{\tau(n)+1} - x^*, Az_{\tau(n)}\rangle$$
$$\leq -2\lambda_{\tau(n)}\langle x_{\tau(n)+1} - x^*, Az_{\tau(n)}\rangle. \qquad (3.47)$$

Which implies that

$$\langle x_{\tau(n)+1} - x^*, Az_{\tau(n)}\rangle \leq 0 \quad \forall_n \geq n_0. \qquad (3.48)$$

Since $\{z_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{z_{\tau(n)}\}$, still denoted by $\{z_{\tau(n)}\}$ which converges weakly to $q \in H$. It is easily checked that $q \in Fix(T)$. Furthermore, we observe that

$$\delta\left\|z_{\tau(n)} - x^*\right\|^2 \leq \langle z_{\tau(n)} - x^*, Az_{\tau(n)} - Ax^*\rangle$$
$$= \langle z_{\tau(n)} - x^* + x_{\tau(n)+1} - x_{\tau(n)+1}, Az_{\tau(n)} - Ax^*\rangle$$
$$= \langle z_{\tau(n)} - x^* + x_{\tau(n)+1} - x_{\tau(n)+1}, Az_{\tau(n)}\rangle + \langle z_{\tau(n)} - x^* + x_{\tau(n)+1} - x_{\tau(n)+1}, -Ax^*\rangle$$
$$= \langle x_{\tau(n)+1} - x^*, Az_{\tau(n)}\rangle + \langle z_{\tau(n)} - x_{\tau(n)+1}, Az_{\tau(n)}\rangle - \langle z_{\tau(n)} - x^*, Ax^*\rangle. \qquad (3.49)$$

Hence, for all $n \geq n_0$,

$$\delta\left\|z_{\tau(n)} - x^*\right\|^2 \leq \langle z_{\tau(n)} - x_{\tau(n)+1}, Az_{\tau(n)}\rangle - \langle z_{\tau(n)} - x^*, Ax^*\rangle.$$

Therefore,From **(2.8)** and **(3.50)**,we have

$$\lim_{n\to\infty} \sup\left\|z_{\tau(n)} - x^*\right\|^2 \leq -\frac{1}{\delta}\langle z_{\tau(n)} - x^*, Ax^*\rangle$$

$$\lim_{n\to\infty} \sup\left\|z_{\tau(n)} - x^*\right\|^2 \leq -\frac{1}{\delta}\langle q - x^*, Ax^*\rangle \leq 0, \tag{3.51}$$

which implies that

$$\lim_{n\to\infty}\left\|z_{\tau(n)} - x^*\right\| = 0. \tag{3.52}$$

Thus,

$$\lim_{n\to\infty}\left\|x_{\tau(n)} - x^*\right\| = 0. \tag{3.53}$$

It is immediate that

$$\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0. \tag{3.54}$$

Furthermore, for $n \geq n_0$, it is easily observed that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (i.e, $\tau(n) < n$), because $\Gamma_j \leq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}. \tag{3.55}$$

Hence $\lim_{n\to\infty}\Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to x^*. Consequently, it easy to prove that $\{y_n\}$ and $\{z_n\}$ converge strongly to x^*. This completes the proof.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] E. Blum andW. Oettli, “Fromoptimization and variational inequalities to equilibrium problems,” *The Mathematics Student*, vol. 63, pp. 123–145, 1994.

[2] L.-C. Zeng, S.-Y. Wu, and J.-C. Yao, “Generalized KKM theorem with applications to generalized minimax inequalities and generalized equilibrium problems,” *Taiwanese Journal of Mathematics*, vol. 10, no. 6, pp. 1497–1514, 2006.

[3] O. Chadli, N. C. Wong, and J.-C. Yao, “Equilibrium problems with applications to eigenvalue problems,” *Journal of Optimization Theory and Applications*, vol. 117, no. 2, pp. 245–266, 2003.

[4] O. Chadli, S. Schaible, and J.-C. Yao, “Regularized equilibrium problems with application to noncoercive hemivariational inequalities,” *Journal of Optimization Theory and Applications*, vol. 121, no. 3, pp. 571–596, 2004.

[5] I. V. Konnov, S. Schaible, and J.-C. Yao, “Combined relaxation method for mixed equilibrium problems,” *Journal of Optimization Theory and Applications*, vol. 126, no. 2, pp. 309–322, 2005.

[6] P. L. Combettes and S. A. Hirstoaga, “Equilibrium programming in Hilbert spaces,” *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005

[7] S. D. Fl°am and A. S. Antipin, “Equilibrium programming using proximal-like algorithms,” *Mathematical Programming*, vol. 78, no. 1, pp. 29–41, 1997.

[8] S. Takahashi and W. Takahashi, “Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces,” *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[9] O. Chadli, I. V. Konnov, and J.-C. Yao, “Descent methods for equilibrium problems in a banach space,” *Computers and Mathematics with Applications*, vol. 48, no. 3-4, pp. 609–616, 2004.

[10] X.-P. Ding, Y.-C. Lin, and J.-C. Yao, “Predictor-corrector algorithms for solving generalized mixed implicit quasi-equilibrium problems,” *AppliedMathematics andMechanics*, vol. 27, no. 9, pp. 1157–1164, 2006.

[11] Y. Yao, Y.-C. Liou, and J.-C. Yao, “Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings,” *Fixed Point Theory and Applications*, vol. 2007, Article ID 64363, 12 pages, 2007.

[12] S. Plubtieng and R. Punpaeng, “A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces,” *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 455– 469, 2007.

[13] A. Tada and W. Takahashi, “Strong convergence theorem for an equilibrium problem and a nonexpansive mapping,” in *Nonlinear Analysis and Convex Analysis*,W. Takahashi and T. Tanaka, Eds., pp. 609–617, Yokohama, Yokohama, Japan, 2007.

[14] M. A. Noor, “Fundamentals of equilibrium problems,” *Mathematical Inequalities & Applications*, vol. 9, no. 3, pp. 529–566, 2006.

[15] Y. Yao, M.A. Noor, and Y-C. Liou, “On iterative methods for equilibrium problems,” *Nonlinear Analysis*, vol. 70, pp. 497–509, 2009.

[16] L.-C. Zeng and J.-C. Yao, “A hybrid iterative scheme for mixed equilibrium problems and fixed point problems,” *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 186–201, 2008.

[17] Y. Yao, M. A. Noor, S. Zainab, and Y.-C. Liou, “Mixed equilibrium problems and optimization problems,” *Journal of Mathematical Analysis and Applications*, vol. 354, no. 1, pp. 319–329, 2009.

[18] Y. Yao, H. Zhou, and Y.-C. Liou, “Weak and strong convergence theorems for an asymptotically *k*strictpseudocontraction and a mixed equilibrium problem,” *Journal of the KoreanMathematical Society*,vol. 46, pp. 561–576, 2009.

[25] G. Marino and H.-K. Xu, "Weak and strong convergence theorems for strict pseudo-contractions inHilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 336–346, 2007.

[26] L.-C. Zeng, N.-C. Wong, and J.-C. Yao, "Strong convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type," *Taiwanese Journal of Mathematics*, vol. 10, no. 4, pp. 837–849, 2006.

[27] H. Zhou, "Convergence theorems of common fixed points for a finite family of Lipschitz pseudocontractions in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 10, pp. 2977–2983, 2008.

[28] I. V. Konnov, "Generalized monotone equilibrium problems and variational inequalities," in *Handbook of Generalized Convexity and Generalized Monotonicity*,N.Hadjisavvas, S. Komlosi, and S. Schaible, Eds.,Springer, New York, NY, USA, 2005.

APPENDIX

BIOGRAPHY

BIOGRAPHY

Name Surnama Nutcha apinyananan

Date of Birth 25 January 1996

Address House 177/2 village No. 6 Inburi,Inburi,

Singburi province 1611

Education Background

Year 2010 Senior High School

Singburi school, Singburi province

Year 2013 Bachelor of Science (Mathematics)

University of Phayao, Phayao province

Name Surnama	Patcharaporn Chaikhod
Date of Birth	14 April 1995
Address	House No. 143 Village No. 6
	Nang Lae, Muang Chiang Rai, Chiang Rai
	Provinc 57100
Education Background	
Year 2010	Senior High School
	Mengraimaharajwitthayakhom, Chiang Rai provincc
Year 2013	Bachelor of Science (Mathematics)
	University of Phayao, Phayao province

Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 632819, 15 pages
doi:10.1155/2009/632819

Research Article

An Extragradient Method for Mixed Equilibrium Problems and Fixed Point Problems

Yonghong Yao,[1] Yeong-Cheng Liou,[2] and Yuh-Jenn Wu[3]

[1] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China*
[2] *Department of Information Management, Cheng Shiu University, Kaohsiung 833, Taiwan*
[3] *Department of Applied Mathematics, Chung Yuan Christian University, Chung Li 320, Taiwan*

Correspondence should be addressed to Yeong-Cheng Liou, simplex_liou@hotmail.com

Received 2 November 2008; Revised 8 April 2009; Accepted 23 May 2009

Recommended by Nan-Jing Huang

The purpose of this paper is to investigate the problem of approximating a common element of the set of fixed points of a demicontractive mapping and the set of solutions of a mixed equilibrium problem. First, we propose an extragradient method for solving the mixed equilibrium problems and the fixed point problems. Subsequently, we prove the strong convergence of the proposed algorithm under some mild assumptions.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\varphi : C \to \mathbf{R}$ be a real-valued function and $\Theta : C \times C \to \mathbf{R}$ be an equilibrium bifunction, that is, $\Theta(u,u) = 0$ for each $u \in C$. We consider the following mixed equilibrium problem (MEP) which is to find $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C. \tag{MEP}$$

In particular, if $\varphi \equiv 0$, this problem reduces to the equilibrium problem (EP), which is to find $x^* \in C$ such that

$$\Theta(x^*, y) \geq 0, \quad \forall y \in C. \tag{EP}$$

Denote the set of solutions of (MEP) by Ω and the set of solutions of (EP) by Γ. The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and the equilibrium problems as special cases; see, for example, [1–5]. Some methods have been proposed to solve the equilibrium problems, see, for example, [5–21].

In 2005, Combettes and Hirstoaga [6] introduced an iterative algorithm of finding the best approximation to the initial data when $\Gamma \neq \emptyset$ and proved a strong convergence theorem. Recently by using the viscosity approximation method S. Takahashi and W. Takahashi [8] introduced another iterative algorithm for finding a common element of the set of solutions of (EP) and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Let $S : C \to H$ be a nonexpansive mapping and $f : C \to C$ be a contraction. Starting with arbitrary initial $x_1 \in H$, define the sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{gathered}\Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,\\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, \quad \forall n \geq 0.\end{gathered} \tag{TT}$$

S. Takahashi and W. Takahashi proved that the sequences $\{x_n\}$ and $\{u_n\}$ defined by (TT) converge strongly to $z \in \mathrm{Fix}(S) \cap \Gamma$ with the following restrictions on algorithm parameters $\{\alpha_n\}$ and $\{r_n\}$:

(i) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $\liminf_{n\to\infty} r_n > 0$;

(iii) (A1): $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$; and (R1): $\sum_{n=0}^{\infty}|r_{n+1} - r_n| < \infty$.

Subsequently, some iterative algorithms for equilibrium problems and fixed point problems have further developed by some authors. In particular, Zeng and Yao [16] introduced a new hybrid iterative algorithm for mixed equilibrium problems and fixed point problems and Mainge and Moudafi [22] introduced an iterative algorithm for equilibrium problems and fixed point problems.

On the other hand, for solving the equilibrium problem (EP), Moudafi [23] presented a new iterative algorithm and proved a weak convergence theorem. Ceng et al. [24] introduced another iterative algorithm for finding an element of $\mathrm{Fix}(S) \cap \Gamma$. Let $S : C \to C$ be a k-strict pseudocontraction for some $0 \leq k < 1$ such that $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. For given $x_1 \in H$, let the sequences $\{x_n\}$ and $\{u_n\}$ be generated iteratively by

$$\begin{gathered}\Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,\\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)Su_n, \quad \forall n \geq 1,\end{gathered} \tag{CAY}$$

where the parameters $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$;

(ii) $\{r_n\} \subset (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (CAY) converge weakly to an element of $\mathrm{Fix}(S) \cap \Gamma$.

At this point, we should point out that all of the above results are interesting and valuable. At the same time, these results also bring us the following conjectures.

Questions

(1) Could we weaken or remove the control condition (iii) on algorithm parameters in S. Takahashi and W. Takahashi [8]?

(2) Could we construct an iterative algorithm for k-strict pseudocontractions such that the strong convergence of the presented algorithm is guaranteed?

(3) Could we give some proof methods which are different from those in [8, 12, 16, 24]?

It is our purpose in this paper that we introduce a general iterative algorithm for approximating a common element of the set of fixed points of a demicontractive mapping and the set of solutions of a mixed equilibrium problem. Subsequently, we prove the strong convergence of the proposed algorithm under some mild assumptions. Our results give positive answers to the above questions.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H.

Let $T : C \to C$ be a mapping. We use $\mathrm{Fix}(T)$ to denote the set of the fixed points of T. Recall what follows.

(i) T is called demicontractive if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - Tx\|^2 \tag{2.1}$$

for all $x \in C$ and $x^* \in \mathrm{Fix}(T)$, which is equivalent to

$$\langle x - Tx, x - x^* \rangle \geq \frac{1-k}{2}\|x - Tx\|^2. \tag{2.2}$$

For such case, we also say that T is a k-demicontractive mapping.

(ii) T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{2.3}$$

for all $x, y \in C$.

(iii) T is called quasi-nonexpansive if

$$\|Tx - x^*\| \leq \|x - x^*\| \tag{2.4}$$

for all $x \in C$ and $x^* \in \mathrm{Fix}(T)$.

(iv) T is called strictly pseudocontractive if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2 \tag{2.5}$$

for all $x, y \in C$.

It is worth noting that the class of demicontractive mappings includes the class of the nonexpansive mappings, the quasi-nonexpansive mappings and the strictly pseudo-contractive mappings as special cases.

Let us also recall that T is called demiclosed if for any sequence $\{x_n\} \subset H$ and $x \in H$, we have

$$x_n \longrightarrow x \text{ weakly}, \quad (I - T)x_n \longrightarrow 0 \text{ strongly} \Longrightarrow x \in \text{Fix}(T). \tag{2.6}$$

It is well-known that the nonexpansive mappings, strictly pseudo-contractive mappings are all demiclosed. See, for example, [25–27].

An operator $A : C \to H$ is said to be δ-strongly monotone if there exists a positive constant δ such that

$$\langle Ax - Ay, x - y \rangle \geq \delta \|x - y\|^2 \tag{2.7}$$

for all $x, y \in C$.

Now we concern the following problem: find $x^* \in \text{Fix}(T) \cap \Omega$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(T) \cap \Omega. \tag{2.8}$$

In this paper, for solving problem (2.8) with an equilibrium bifunction $\Theta : C \times C \to \mathbf{R}$, we assume that Θ satisfies the following conditions:

(H1) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$;

(H2) for each fixed $y \in C$, $x \mapsto \Theta(x, y)$ is concave and upper semicontinuous;

(H3) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex.

A mapping $\eta : C \times C \to H$ is called Lipschitz continuous, if there exists a constant $\lambda > 0$ such that

$$\|\eta(x, y)\| \leq \lambda \|x - y\|, \quad \forall x, y \in C. \tag{2.9}$$

A differentiable function $K : C \to \mathbf{R}$ on a convex set C is called

(i) η-convex if

$$K(y) - K(x) \geq \langle K'(x), \eta(y, x) \rangle, \quad \forall x, y \in C, \tag{2.10}$$

where K' is the Frechet derivative of K at x;

(ii) η-strongly convex if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \langle K'(x), \eta(y, x) \rangle \geq \left(\frac{\sigma}{2}\right) \|x - y\|^2, \quad \forall x, y \in C. \tag{2.11}$$

Let C be a nonempty closed convex subset of a real Hilbert space H, $\varphi : C \to \mathbf{R}$ be real-valued function and $\Theta : C \times C \to \mathbf{R}$ be an equilibrium bifunction. Let r be a positive number. For a given point $x \in C$, the auxiliary problem for (MEP) consists of finding $y \in C$ such that

$$\Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z,y)\rangle \geq 0, \quad \forall z \in C. \tag{2.12}$$

Let $S_r : C \to C$ be the mapping such that for each $x \in C$, $S_r(x)$ is the solution set of the auxiliary problem, that is, $\forall x \in C$,

$$S_r(x) = \left\{ y \in C : \Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z,y)\rangle \geq 0, \forall z \in C \right\}. \tag{2.13}$$

We need the following important and interesting result for proving our main results.

Lemma 2.1 ([16, 28]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to \mathbf{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3). Assume what follows.*

(i) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that*

  (a) *$\eta(x,y) + \eta(y,x) = 0$, $\forall x, y \in C$,*
  (b) *$\eta(\cdot,\cdot)$ is affine in the first variable,*
  (c) *for each fixed $y \in C$, $x \mapsto \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.*

(ii) *$K : C \to \mathbf{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is sequentially continuous from the weak topology to the strong topology.*

(iii) *For each $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,*

$$\Theta(y,z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x,y)\rangle < 0. \tag{2.14}$$

Then there hold the following:

(i) *S_r is single-valued;*

(ii) *S_r is nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$ and*

$$\left\langle K'(x_1) - K'(x_2), \eta(u_1,u_2)\right\rangle \geq \left\langle K'(u_1) - K'(u_2), \eta(u_1,u_2)\right\rangle, \quad \forall (x_1,x_2) \in C \times C, \tag{2.15}$$

where $u_i = S_r(x_i)$ for $i = 1,2$;

(iii) $\mathrm{Fix}(S_r) = \Omega$;

(iv) *Ω is closed and convex.*

3. Main Results

Let H be a real Hilbert space, $\varphi : H \to \mathbf{R}$ be a lower semicontinuous and convex real-valued function, $\Theta : H \times H \to \mathbf{R}$ be an equilibrium bifunction. Let $A : H \to H$ be a mapping and $T : H \to H$ be a mapping. In this section, we first introduce the following new iterative algorithm.

Algorithm 3.1. Let r be a positive parameter. Let $\{\lambda_n\}$ be a sequence in $[0, \infty)$ and $\{\alpha_n\}$ be a sequence in $[0, 1)$. Define the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ by the following manner:

$$\begin{gathered} x_0 \in C \text{ chosen arbitrarily,} \\ \Theta(z_n, x) + \varphi(x) - \varphi(z_n) + \frac{1}{r}\langle K'(z_n) - K'(x_n), \eta(x, z_n)\rangle \geq 0, \quad \forall x \in C, \\ y_n = z_n - \lambda_n A z_n, \\ x_{n+1} = (1 - \alpha_n) y_n + \alpha_n T y_n. \end{gathered} \tag{3.1}$$

Now we give a strong convergence result concerning Algorithm 3.1 as follows.

Theorem 3.2. *Let H be a real Hilbert space. Let $\varphi : H \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $\Theta : H \times H \to \mathbf{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3). Let $A : H \to H$ be an L-Lipschitz continuous and δ-strongly monotone mapping and $T : H \to H$ be a demiclosed and k-demicontractive mapping such that* $\mathrm{Fix}(T) \cap \Omega \neq \emptyset$. *Assume what follows.*

(i) $\eta : H \times H \to H$ *is Lipschitz continuous with constant $\lambda > 0$ such that*

  (a) $\eta(x, y) + \eta(y, x) = 0$, $\forall x, y \in H$,

  (b) $\eta(\cdot, \cdot)$ *is affine in the first variable,*

  (c) *for each fixed $y \in H$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology.*

(ii) $K : H \to \mathbf{R}$ *is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$.*

(iii) *For each $x \in H$; there exist a bounded subset $D_x \subset H$ and $z_x \in H$ such that, for any $y \notin D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0. \tag{3.2}$$

(iv) $\alpha_n \in [\gamma, (1-k)/2]$ *for some* $\gamma > 0$, $\lim_{n\to\infty}\lambda_n = 0$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated by (3.1) *converge strongly to x^* which solves the problem* (2.8) *provided S_r is firmly nonexpansive.*

Proof. First, we prove that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are all bounded. Without loss of generality, we may assume that $0 < \delta < L$. Given $\mu \in (0, 2\delta/L^2)$ and $x, y \in H$, we have

$$\begin{aligned}\|(\mu A - I)x - (\mu A - I)y\|^2 &= \mu^2\|Ax - Ay\|^2 + \|x - y\|^2 - 2\mu\langle Ax - Ay, x - y\rangle \\ &\leq \mu^2 L^2\|x - y\|^2 + \|x - y\|^2 - 2\mu\delta\|x - y\|^2 \\ &= \left(1 - 2\mu\delta + \mu^2 L^2\right)\|x - y\|^2,\end{aligned} \tag{3.3}$$

that is,

$$\|(\mu A - I)x - (\mu A - I)y\| \leq \sqrt{1 - 2\mu\delta + \mu^2 L^2}\|x - y\|. \tag{3.4}$$

Take $x^* \in \operatorname{Fix}(T) \cap \Omega$. From (3.1), we have

$$\begin{aligned}\|y_{n+1} - (x^* - \lambda_{n+1}Ax^*)\| &= \|(z_{n+1} - \lambda_{n+1}Az_{n+1}) - (x^* - \lambda_{n+1}Ax^*)\| \\ &= \left\|\left(1 - \frac{\lambda_{n+1}}{\mu}\right)(z_{n+1} - x^*) - \frac{\lambda_{n+1}}{\mu}\left((\mu A - I)z_{n+1} - (\mu A - I)x^*\right)\right\| \\ &\leq \left(1 - \frac{\lambda_{n+1}}{\mu}\right)\|z_{n+1} - x^*\| + \frac{\lambda_{n+1}}{\mu}\|(\mu A - I)z_{n+1} - (\mu A - I)x^*\|.\end{aligned} \tag{3.5}$$

Therefore,

$$\|y_{n+1} - (x^* - \lambda_{n+1}Ax^*)\| \leq \left(1 - \frac{\lambda_{n+1}\omega}{\mu}\right)\|z_{n+1} - x^*\|, \tag{3.6}$$

where $\omega = 1 - \sqrt{1 - 2\mu\delta + \mu^2 L^2} \in (0,1)$.

Note that $z_{n+1} = S_r x_{n+1}$ and S_r are firmly nonexpansive. Hence, we have

$$\begin{aligned}\|z_{n+1} - x^*\|^2 &= \|S_r x_{n+1} - S_r x^*\|^2 \\ &\leq \langle S_r x_{n+1} - S_r x^*, x_{n+1} - x^*\rangle \\ &= \langle z_{n+1} - x^*, x_{n+1} - x^*\rangle \\ &= \frac{1}{2}\left(\|z_{n+1} - x^*\|^2 + \|x_{n+1} - x^*\|^2 - \|x_{n+1} - z_{n+1}\|^2\right),\end{aligned} \tag{3.7}$$

which implies that

$$\|z_{n+1} - x^*\|^2 \leq \|x_{n+1} - x^*\|^2 - \|x_{n+1} - z_{n+1}\|^2. \tag{3.8}$$

From (2.2) and (3.1), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|(1-\alpha_n)y_n+\alpha_n T y_n - x^*\|^2 \\
&= \|(y_n-x^*)-\alpha_n(y_n-Ty_n)\|^2 \\
&= \|y_n-x^*\|^2 - 2\alpha_n\langle y_n-Ty_n, y_n-x^*\rangle + \alpha_n^2\|y_n-Ty_n\|^2 \\
&\le \|y_n-x^*\|^2 - 2\alpha_n\frac{1-k}{2}\|y_n-Ty_n\|^2 + \alpha_n^2\|y_n-Ty_n\|^2 \\
&= \|y_n-x^*\|^2 - \alpha_n(1-k-\alpha_n)\|y_n-Ty_n\|^2 \\
&\le \|y_n-x^*\|^2.
\end{aligned} \tag{3.9}$$

From (3.6)–(3.9), we have

$$\begin{aligned}
\|y_{n+1}-x^*\| &\le \|y_{n+1}-(x^*-\lambda_{n+1}Ax^*)\| + \lambda_{n+1}\|Ax^*\| \\
&\le \left(1-\frac{\lambda_{n+1}\omega}{\mu}\right)\|z_{n+1}-x^*\| + \lambda_{n+1}\|Ax^*\| \\
&\le \left(1-\frac{\lambda_{n+1}\omega}{\mu}\right)\|x_{n+1}-x^*\| + \lambda_{n+1}\|Ax^*\| \\
&\le \left(1-\frac{\lambda_{n+1}\omega}{\mu}\right)\|y_n-x^*\| + \lambda_{n+1}\|Ax^*\| \\
&= \left(1-\frac{\lambda_{n+1}\omega}{\mu}\right)\|y_n-x^*\| + \frac{\lambda_{n+1}\omega}{\mu}\left\{\frac{\mu}{\omega}\|Ax^*\|\right\} \\
&\le \max\left\{\|y_n-x^*\|, \frac{\mu\|Ax^*\|}{\omega}\right\} \\
&\le \cdots \\
&\le \max\left\{\|y_0-x^*\|, \frac{\mu\|Ax^*\|}{\omega}\right\}.
\end{aligned} \tag{3.10}$$

This implies that $\{y_n\}$ is bounded, so are $\{x_n\}$ and $\{z_n\}$.

From (3.1), we can write $y_n - Ty_n = (1/\alpha_n)(y_n - x_{n+1})$. Thus, from (3.9), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \|y_n-x^*\|^2 - \alpha_n(1-k-\alpha_n)\|y_n-Ty_n\|^2 \\
&\le \|y_n-x^*\|^2 - \frac{1-k-\alpha_n}{\alpha_n}\|y_n-x_{n+1}\|^2.
\end{aligned} \tag{3.11}$$

Since $\alpha_n \in (0, (1-k)/2]$, $(1-k-\alpha_n)/\alpha_n \geq 1$. Therefore, from (3.8) and (3.11), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|y_n - x^*\|^2 - \|y_n - x_{n+1}\|^2 \\
&= \|z_n - x^* - \lambda_n A z_n\|^2 - \|z_n - x_{n+1} - \lambda_n A z_n\|^2 \\
&= \|z_n - x^*\|^2 - 2\lambda_n \langle A z_n, z_n - x^* \rangle + \lambda_n^2 \|A z_n\|^2 \\
&\quad - \|z_n - x_{n+1}\|^2 + 2\lambda_n \langle A z_n, z_n - x_{n+1} \rangle - \lambda_n^2 \|A z_n\|^2 \\
&= \|z_n - x^*\|^2 - 2\lambda_n \langle x_{n+1} - x^*, A z_n \rangle - \|x_{n+1} - z_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 - 2\lambda_n \langle x_{n+1} - x^*, A z_n \rangle - \|x_{n+1} - z_n\|^2.
\end{aligned} \tag{3.12}$$

We note that $\{x_n\}$ and $\{z_n\}$ are bounded. So there exists a constant $M \geq 0$ such that

$$|\langle x_{n+1} - x^*, A z_n \rangle| \leq M \quad \forall n \geq 0. \tag{3.13}$$

Consequently, we get

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 + \|x_{n+1} - z_n\|^2 + \|x_n - z_n\|^2 \leq 2M\lambda_n. \tag{3.14}$$

Now we divide two cases to prove that $\{x_n\}$ converges strongly to x^*.

Case 1. Assume that the sequence $\{\|x_n - x^*\|\}$ is a monotone sequence. Then $\{\|x_n - x^*\|\}$ is convergent. Setting $\lim_{n\to\infty}\|x_n - x^*\| = d$.

(i) If $d = 0$, then the desired conclusion is obtained.

(ii) Assume that $d > 0$. Clearly, we have

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 \longrightarrow 0, \tag{3.15}$$

this together with $\lambda_n \to 0$ and (3.14) implies that

$$\|x_{n+1} - z_n\|^2 + \|x_n - z_n\|^2 \longrightarrow 0, \tag{3.16}$$

that is to say

$$\|x_{n+1} - z_n\| \longrightarrow 0, \qquad \|x_n - z_n\| \longrightarrow 0. \tag{3.17}$$

Let $z \in H$ be a weak limit point of $\{z_{n_k}\}$. Then there exists a subsequence of $\{z_{n_k}\}$, still denoted by $\{z_{n_k}\}$ which weakly converges to z. Noting that $\lambda_n \to 0$, we also have

$$y_{n_k} = z_{n_k} - \lambda_{n_k} A z_{n_k} \longrightarrow z \text{ weakly}. \tag{3.18}$$

Combining (3.1) and (3.17), we have

$$\begin{aligned}\|Ty_{n_k}-y_{n_k}\| &= \frac{1}{\alpha_{n_k}}\|y_{n_k}-x_{n_k+1}\| \\ &= \frac{1}{\alpha_{n_k}}\|x_{n_k+1}-z_{n_k}+\lambda_{n_k}Az_{n_k}\| \\ &\le \|x_{n_k+1}-z_{n_k}\|+\lambda_{n_k}\|Az_{n_k}\| \\ &\longrightarrow 0.\end{aligned} \tag{3.19}$$

Since T is demiclosed, then we obtain $z\in \mathrm{Fix}(T)$.

Next we show that $z\in\Omega$. Since $z_n=S_rx_n$, we derive

$$\Theta(z_n,x)+\varphi(x)-\varphi(z_n)+\frac{1}{r}\langle K'(z_n)-K'(x_n),\eta(x,z_n)\rangle\ge 0,\quad \forall x\in C. \tag{3.20}$$

From the monotonicity of Θ, we have

$$\frac{1}{r}\langle K'(z_n)-K'(x_n),\eta(x,z_n)\rangle+\varphi(x)-\varphi(z_n)\ge -\Theta(z_n,x)\ge\Theta(x,z_n), \tag{3.21}$$

and hence

$$\left\langle \frac{K'(z_{n_k})-K'(x_{n_k})}{r},\eta(x,z_{n_k})\right\rangle+\varphi(x)-\varphi(z_{n_k})\ge\Theta(x,z_{n_k}). \tag{3.22}$$

Since $(K'(z_{n_k})-K'(x_{n_k}))/r\to 0$ and $z_{n_k}\to z$ weakly, from the weak lower semicontinuity of φ and $\Theta(x,y)$ in the second variable y, we have

$$\Theta(x,z)+\varphi(z)-\varphi(x)\le 0, \tag{3.23}$$

for all $x\in C$. For $0<t\le 1$ and $x\in C$, let $x_t=tx+(1-t)z$. Since $x\in C$ and $z\in C$, we have $x_t\in C$ and hence $\Theta(x_t,z)+\varphi(z)-\varphi(x_t)\le 0$. From the convexity of equilibrium bifunction $\Theta(x,y)$ in the second variable y, we have

$$\begin{aligned}0&=\Theta(x_t,x_t)+\varphi(x_t)-\varphi(x_t)\\ &\le t\Theta(x_t,x)+(1-t)\Theta(x_t,z)+t\varphi(x)+(1-t)\varphi(z)-\varphi(x_t)\\ &\le t\left[\Theta(x_t,x)+\varphi(x)-\varphi(x_t)\right],\end{aligned} \tag{3.24}$$

and hence $\Theta(x_t,x)+\varphi(x)-\varphi(x_t)\ge 0$. Then, we have

$$\Theta(z,x)+\varphi(x)-\varphi(z)\ge 0 \tag{3.25}$$

for all $x\in C$ and hence $z\in\Omega$.

Therefore, we have

$$z \in \operatorname{Fix}(T) \cap \Omega. \tag{3.26}$$

Thus, if x^* is a solution of problem (2.8), we have

$$\liminf_{k \to \infty} \langle z_{n_k} - x^*, Ax^* \rangle = \langle z - x^*, Ax^* \rangle \geq 0. \tag{3.27}$$

Suppose that there exists another subsequence $\{z_{n_i}\}$ which weakly converges to z'. It is easily checked that $z' \in \operatorname{Fix}(T) \cap \Omega$ and

$$\liminf_{i \to \infty} \langle z_{n_i} - x^*, Ax^* \rangle = \langle z' - x^*, Ax^* \rangle \geq 0. \tag{3.28}$$

Therefor, we have

$$\liminf_{n \to \infty} \langle z_n - x^*, Ax^* \rangle \geq 0. \tag{3.29}$$

Since A is δ-strongly monotone, we have

$$\langle x_{n+1} - x^*, Az_n \rangle \geq \delta \|z_n - x^*\|^2 + \langle z_n - x^*, Ax^* \rangle + \langle x_{n+1} - z_n, Az_n \rangle. \tag{3.30}$$

By (3.17)–(3.30), we get

$$\liminf_{n \to \infty} \langle x_{n+1} - x^*, Az_n \rangle \geq \delta d^2. \tag{3.31}$$

From (3.12), for $0 < \epsilon < \delta d^2$, we deduce that there exists a positive integer number n_0 large enough, when $n \geq n_0$,

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 \leq -2\lambda_n \left(\delta d^2 - \epsilon \right). \tag{3.32}$$

This implies that

$$\|x_{n+1} - x^*\|^2 - \|x_{n_0} - x^*\|^2 \leq -2\left(\delta d^2 - \epsilon \right) \sum_{k=n_0}^{n} \lambda_k. \tag{3.33}$$

Since $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\{x_n\}$ is bounded, hence the last inequality is a contraction. Therefore, $d = 0$, that is to say, $x_n \to x^*$.

Case 2. Assume that $\{\|x_n - x^*\|\}$ is not a monotone sequence. Set $\Gamma_n = \|x_n - x^*\|^2$ and let $\tau : N \to N$ be a mapping for all $n \geq n_0$ by

$$\tau(n) = \max\{k \in N : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}. \tag{3.34}$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \longrightarrow \infty$ as $n \longrightarrow \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$. From (3.14), we have

$$\|x_{\tau(n)+1} - z_{\tau(n)}\|^2 + \|x_{\tau(n)} - z_{\tau(n)}\|^2 \leq 2M\lambda_{\tau(n)} \longrightarrow 0, \tag{3.35}$$

thus

$$\|x_{\tau(n)+1} - z_{\tau(n)}\| \longrightarrow 0, \qquad \|x_{\tau(n)} - z_{\tau(n)}\| \longrightarrow 0. \tag{3.36}$$

Therefore,

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \longrightarrow 0. \tag{3.37}$$

Since $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, for all $n \geq n_0$, from (3.12), we get

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 + \|x_{\tau(n)+1} - z_{\tau(n)}\|^2 + \|x_{\tau(n)} - z_{\tau(n)}\|^2 \\ &\leq -2\lambda_{\tau(n)}\left\langle x_{\tau(n)+1} - x^*, Az_{\tau(n)}\right\rangle, \end{aligned} \tag{3.38}$$

which implies that

$$\left\langle x_{\tau(n)+1} - x^*, Az_{\tau(n)}\right\rangle \leq 0 \quad \forall n \geq n_0. \tag{3.39}$$

Since $\{z_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{z_{\tau(n)}\}$, still denoted by $\{z_{\tau(n)}\}$ which converges weakly to $q \in H$. It is easily checked that $q \in \operatorname{Fix}(T) \cap \Omega$. Furthermore, we observe that

$$\begin{aligned} \delta\|z_{\tau(n)} - x^*\|^2 &\leq \left\langle z_{\tau(n)} - x^*, Az_{\tau(n)} - Ax^*\right\rangle \\ &= \left\langle x_{\tau(n)+1} - x^*, Az_{\tau(n)}\right\rangle + \left\langle z_{\tau(n)} - x_{\tau(n)+1}, Az_{\tau(n)}\right\rangle - \left\langle z_{\tau(n)} - x^*, Ax^*\right\rangle. \end{aligned} \tag{3.40}$$

Hence, for all $n \geq n_0$,

$$\delta\|z_{\tau(n)} - x^*\|^2 \leq \left\langle z_{\tau(n)} - x_{\tau(n)+1}, Az_{\tau(n)}\right\rangle - \left\langle z_{\tau(n)} - x^*, Ax^*\right\rangle. \tag{3.41}$$

Therefore

$$\limsup_{n\to\infty}\|z_{\tau(n)} - x^*\|^2 \leq -\frac{1}{\delta}\left\langle q - x^*, Ax^*\right\rangle \leq 0, \tag{3.42}$$

which implies that

$$\lim_{n\to\infty}\|z_{\tau(n)} - x^*\| = 0. \tag{3.43}$$

Thus,

$$\lim_{n\to\infty} \|x_{\tau(n)} - x^*\| = 0. \tag{3.44}$$

It is immediate that

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0. \tag{3.45}$$

Furthermore, for $n \ge n_0$, it is easily observed that $\Gamma_n \le \Gamma_{\tau(n)+1}$ if $n \ne \tau(n)$ (i.e., $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \le j \le n$. As a consequence, we obtain for all $n \ge n_0$,

$$0 \le \Gamma_n \le \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}. \tag{3.46}$$

Hence $\lim_{n\to\infty}\Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to x^*. Consequently, it easy to prove that $\{y_n\}$ and $\{z_n\}$ converge strongly to x^*. This completes the proof. □

Remark 3.3. The advantages of these results in this paper are that less restrictions on the parameters $\{\lambda_n\}$ are imposed.

As direct consequence of Theorem 3.2, we obtain the following.

Corollary 3.4. *Let H be a real Hilbert space. Let $\varphi : H \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $\Theta : H \times H \to \mathbf{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3). Let $A : H \to H$ be an L-Lipschitz continuous and δ-strongly monotone mapping and $T : H \to H$ be a nonexpansive mapping such that* $\mathrm{Fix}(T) \cap \Omega \ne \emptyset$. *Assume what follows.*

(i) *$\eta : H \times H \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that;*

  (a) *$\eta(x, y) + \eta(y, x) = 0$, $\forall x, y \in H$,*
  (b) *$\eta(\cdot,\cdot)$ is affine in the first variable,*
  (c) *for each fixed $y \in H$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology.*

(ii) *$K : H \to \mathbf{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \ge \lambda\nu$.*

(iii) *For each $x \in H$; there exist a bounded subset $D_x \subset H$ and $z_x \in H$ such that, for any $y \notin D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0. \tag{3.47}$$

(iv) *$\alpha_n \in [\gamma, (1-k)/2]$ for some $\gamma > 0$, $\lim_{n\to\infty}\lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$.*

Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ generated by (3.1) *converge strongly to x^* which solves the problem* (2.8) *provided S_r is firmly nonexpansive.*

Corollary 3.5. *Let H be a real Hilbert space. Let $\varphi : H \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $\Theta : H \times H \to \mathbf{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H3). Let $A : H \to H$ be an L-Lipschitz continuous and δ-strongly monotone mapping and $T : H \to H$ be a strictly pseudo-contractive mapping such that* $\operatorname{Fix}(T) \cap \Omega \neq \emptyset$. *Assume what follows.*

(i) $\eta : H \times H \to H$ *is Lipschitz continuous with constant $\lambda > 0$ such that*

(a) $\eta(x, y) + \eta(y, x) = 0$, $\forall x, y \in H$,

(b) $\eta(\cdot, \cdot)$ *is affine in the first variable,*

(c) *for each fixed $y \in H$, $x \mapsto \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology.*

(ii) $K : H \to \mathbf{R}$ *is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$.*

(iii) *For each $x \in H$; there exist a bounded subset $D_x \subset H$ and $z_x \in H$ such that, for any $y \notin D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0. \tag{3.48}$$

(iv) $\alpha_n \in [\gamma, (1-k)/2]$ *for some $\gamma > 0$, $\lim_{n\to\infty}\lambda_n = 0$ and $\sum_{n=0}^{\infty}\lambda_n = \infty$.*

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by (3.1) *converge strongly to x^* which solves the problem* (2.8) *provided S_r is firmly nonexpansive.*

Acknowledgments

The authors are extremely grateful to the anonymous referee for his/her useful comments and suggestions. The first author was partially supposed by National Natural Science Foundation of China Grant 10771050. The second author was partially supposed by the Grant NSC 97-2221-E-230-017.

References

[1] E. Blum and W. Oettli, “From optimization and variational inequalities to equilibrium problems,” *The Mathematics Student*, vol. 63, pp. 123–145, 1994.

[2] L.-C. Zeng, S.-Y. Wu, and J.-C. Yao, “Generalized KKM theorem with applications to generalized minimax inequalities and generalized equilibrium problems,” *Taiwanese Journal of Mathematics*, vol. 10, no. 6, pp. 1497–1514, 2006.

[3] O. Chadli, N. C. Wong, and J.-C. Yao, “Equilibrium problems with applications to eigenvalue problems,” *Journal of Optimization Theory and Applications*, vol. 117, no. 2, pp. 245–266, 2003.

[4] O. Chadli, S. Schaible, and J.-C. Yao, “Regularized equilibrium problems with application to noncoercive hemivariational inequalities,” *Journal of Optimization Theory and Applications*, vol. 121, no. 3, pp. 571–596, 2004.

[5] I. V. Konnov, S. Schaible, and J.-C. Yao, “Combined relaxation method for mixed equilibrium problems,” *Journal of Optimization Theory and Applications*, vol. 126, no. 2, pp. 309–322, 2005.

[6] P. L. Combettes and S. A. Hirstoaga, “Equilibrium programming in Hilbert spaces,” *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.

[7] S. D. Flåm and A. S. Antipin, "Equilibrium programming using proximal-like algorithms," *Mathematical Programming*, vol. 78, no. 1, pp. 29–41, 1997.

[8] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[9] O. Chadli, I. V. Konnov, and J.-C. Yao, "Descent methods for equilibrium problems in a banach space," *Computers and Mathematics with Applications*, vol. 48, no. 3-4, pp. 609–616, 2004.

[10] X.-P. Ding, Y.-C. Lin, and J.-C. Yao, "Predictor-corrector algorithms for solving generalized mixed implicit quasi-equilibrium problems," *Applied Mathematics and Mechanics*, vol. 27, no. 9, pp. 1157–1164, 2006.

[11] Y. Yao, Y.-C. Liou, and J.-C. Yao, "Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2007, Article ID 64363, 12 pages, 2007.

[12] S. Plubtieng and R. Punpaeng, "A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 455–469, 2007.

[13] A. Tada and W. Takahashi, "Strong convergence theorem for an equilibrium problem and a nonexpansive mapping," in *Nonlinear Analysis and Convex Analysis*, W. Takahashi and T. Tanaka, Eds., pp. 609–617, Yokohama, Yokohama, Japan, 2007.

[14] M. A. Noor, "Fundamentals of equilibrium problems," *Mathematical Inequalities & Applications*, vol. 9, no. 3, pp. 529–566, 2006.

[15] Y. Yao, M.A. Noor, and Y-C. Liou, "On iterative methods for equilibrium problems," *Nonlinear Analysis*, vol. 70, pp. 497–509, 2009.

[16] L.-C. Zeng and J.-C. Yao, "A hybrid iterative scheme for mixed equilibrium problems and fixed point problems," *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 186–201, 2008.

[17] Y. Yao, M. A. Noor, S. Zainab, and Y.-C. Liou, "Mixed equilibrium problems and optimization problems," *Journal of Mathematical Analysis and Applications*, vol. 354, no. 1, pp. 319–329, 2009.

[18] Y. Yao, H. Zhou, and Y.-C. Liou, "Weak and strong convergence theorems for an asymptotically k-strict pseudocontraction and a mixed equilibrium problem," *Journal of the Korean Mathematical Society*, vol. 46, pp. 561–576, 2009.

[19] P.-E. Mainge, "Regularized and inertial algorithms for common fixed points of nonlinear operators," *Journal of Mathematical Analysis and Applications*, vol. 344, no. 2, pp. 876–887, 2008.

[20] Y. Yao, Y.-C. Liou, and J.-C. Yao, "An iterative algorithm for approximating convex minimization problem," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 648–656, 2007.

[21] G. Marino, V. Colao, L. Muglia, and Y. Yao, "Krasnoselski-Mann iteration for hierarchical fixed points and equilibrium problem," *Bulletin of the Australian Mathematical Society*, vol. 79, pp. 187–200, 2009.

[22] P.-E. Mainge and A. Moudafi, "Coupling viscosity methods with the extragradient algorithm for solving equilibrium problems," *Journal of Nonlinear and Convex Analysis*, vol. 9, no. 2, pp. 283–294, 2008.

[23] A. Moudafi, "Weak convergence theorems for nonexpansive mappings and equilibrium problems," *Journal of Nonlinear and Convex Analysis*, vol. 9, no. 1, pp. 37–43, 2008.

[24] L.-C. Ceng, S. Al-Homidan, Q. H. Ansari, and J.-C. Yao, "An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings," *Journal of Computational and Applied Mathematics*, vol. 223, no. 2, pp. 967–974, 2009.

[25] G. Marino and H.-K. Xu, "Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 336–346, 2007.

[26] L.-C. Zeng, N.-C. Wong, and J.-C. Yao, "Strong convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type," *Taiwanese Journal of Mathematics*, vol. 10, no. 4, pp. 837–849, 2006.

[27] H. Zhou, "Convergence theorems of common fixed points for a finite family of Lipschitz pseudocontractions in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 10, pp. 2977–2983, 2008.

[28] I. V. Konnov, "Generalized monotone equilibrium problems and variational inequalities," in *Handbook of Generalized Convexity and Generalized Monotonicity*, N. Hadjisavvas, S. Komlosi, and S. Schaible, Eds., Springer, New York, NY, USA, 2005.