BIPOLAR FUZZY UP-ALGEBRAS

KORAWUT KAWILA
CHAIPHON UDOMSETCHAI

An Independent Study Submitted in Partial Fulfillment

of the Requirements for the degree of Bachelor

of Science Program in Mathematics

April 2018

University of Phayao

BIPOLAR FUZZY UP-ALGEBRAS

KORAWUT KAWILA
CHAIPHON UDOMSETCHAI

An Independent Study Submitted in Partial Fulfillment

of the Requirements for the degree of Bachelor

of Science Program in Mathematics

April 2018

University of Phayao

Advisor and Dean of School of Science have considered the independent study entitled "Bipolar Fuzzy UP-Algebras" submitted in partial fulfillment of the requirements for the degree of Bachelor of Science Program in Mathematics is hereby approved

....

(Assoc. Prof. Dr. Tanakit Thianwan)

Chairman

....

(Assist. Prof. Dr. Aiyared Iampan)

Committee and Advisor

....

(Lect. Peerapong Suebsan)

Committee

....

(Assoc. Prof. Preeyanan Sanpote)

Dean of School of Science

April 2018

ACKNOWLEDGEMENTS

This independent study could not successfully completed without the kindness of advisor, Assistant Professor Dr. Aiyared Iampan for his timely advice, guidance, invaluable help and constant encouragement throughout the course of this independent study since start until successfully. So, We would like to express our special thanks to him.

In addition, we are grateful for all teachers who had provided various kinds of knowledge as well.

Finally, We wish to thank our parents and friends for all their support and encouragement throughout the period of this independent study.

Korawut Kawila

Chaiphon Udomsetchai

ชื่อเรื่อง	พีชคณิตยูพีวิภัชนัยสองขั้ว
ผู้ศึกษาค้นคว้า	นายกรวุฒิ กาวิละ นายชัยพล อุดมเสฏฐ์ชัย
อาจารย์ที่ปรึกษา	ผู้ช่วยศาสตราจารย์ ดร.อัยเรศ เอี่ยมพันธ์
วิทยาศาสตรบัณฑิต	สาขาวิชาคณิตศาสตร์
คำสำคัญ	พีชคณิตยูพี พีชคณิตย่อยยูพีวิภัชนัยสองขั้ว ตัวกรองยูพีวิภัชนัยสองขั้ว ไอดีลยูพีวิภัชนัยสองขั้ว ไอดีลยูพีอย่างเข้มวิภัชนัยสองขั้ว

บทคัดย่อ

ในงานวิจัยนี้ เรานำแนวคิดของเซตวิภัชนัยสองขั้วค่ามาใช้กับพีชคณิตยูพี เราแนะนำแนวคิดของพีชคณิตย่อยยูพีวิภัชนัยสองขั้ว (ตัวกรองยูพีวิภัชนัยสองขั้ว ไอดีลยูพีวิภัชนัยสองขั้ว ไอดีลยูพีอย่างเข้มวิภัชนัยสองขั้ว ตามลำดับ) ของพีชคณิตยูพี และพิสูจน์การวางนัยทั่วไปของแนวคิดข้างต้น และหาเงื่อนไขสำหรับตัวกรองยูพีวิภัชนัยสองขั้วเป็นไอดีลยูพีวิภัชนัยสองขั้ว นอกจากนี้ เราศึกษาความสัมพันธ์ระหว่างพีชคณิตย่อยยูพีวิภัชนัยสองขั้ว (ตัวกรองยูพีวิภัชนัยสองขั้ว ไอดีลยูพีวิภัชนัยสองขั้ว ไอดีลยูพีอย่างเข้มวิภัชนัยสองขั้ว ตามลำดับ) และส่วนตัดระดับของเซตวิภัชนัยสองขั้วข้างต้น

Title	Bipolar Fuzzy UP-Algebras
Author	Mr. Korawut Kawila
	Mr. Chaiphon Udomsetchai
Advisor	Assistant Professor Dr. Aiyared Iampan
Bachelor of Science	Program in Mathematics
Keywords	UP-algebra, bipolar fuzzy UP-subalgebra,
	bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal,
	bipolar fuzzy strongly UP-ideal

ABSTRACT

In this research, we apply the notion of bipolar-valued fuzzy set to UP-algebras. We introduce the notions of bipolar fuzzy UP-subalgebras (resp., bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals and bipolar fuzzy strongly UP-ideals) of UP-algebras and prove its generalizations. We provide a condition for a bipolar fuzzy UP-filter to be a bipolar fuzzy UP-ideal. Further, we discuss the relation between bipolar fuzzy UP-subalgebras (resp., bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals and bipolar fuzzy strongly UP-ideals) and its level cuts.

LIST OF CONTENTS

CHAPTER 1
Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [4], BCI-algebras [5], BCH-algebras [2], KU-algebras [13], SU-algebras [9], UP-algebras [3] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [5] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [4, 5] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

The notion of fuzzy subsets of a set was first considered by Zadeh [19] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, bipolar-valued fuzzy sets etc. The notion of bipolar-valued fuzzy sets was first introduced by Lee [11] in 2000, is an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 0]$.

After the introduction of the notion of bipolar-valued fuzzy sets by Lee [11], several researches were conducted on the generalizations of the notion of bipolar-valued fuzzy sets and application to many logical algebras such as: In 2008, Jun and Song [8] introduced the notions of bipolar fuzzy subalgebras and bipolar fuzzy closed ideals in BCH-algebras. In 2009, Jun and Park [7] introduced the notions of bipolar fuzzy regularities, bipolar fuzzy regular subalgebras, bipolar fuzzy filters, and bipolar fuzzy closed quasi filters in BCH-algebras. In 2011, Lee and Jun [10] introduced the notion of bipolar fuzzy a-ideals of BCI-algebras. In 2012, Jun et al. [6] introduced the notions of bipolar fuzzy CI-subalgebras, bipolar fuzzy ideals and (closed) bipolar fuzzy filters in CI-algebras. In 2014, Muhiuddin

[12] introduced the notions of bipolar fuzzy KU-subalgebras and bipolar fuzzy KU-ideals in KU-algebras. In 2015, Senapati [17] introduced the notion of bipolar fuzzy BG-subalgebras in BG-algebras. In 2016, Sabarinathan et al. [15] introduced the notion of bipolar valued fuzzy ideals of BF-algebras. Sabarinathan et al. [14] introduced the notion of bipolar valued fuzzy α-ideals of BF-algebras. In 2017, Sabarinathan et al. [16] introduced the notion of bipolar valued fuzzy H-ideals of BF-algebras.

In this paper, we apply the notion of bipolar-valued fuzzy set to UP-algebras. We introduce the notions of bipolar fuzzy UP-subalgebras (resp., bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals and bipolar fuzzy strongly UP-ideals) of UP-algebras and prove its generalizations. We provide a condition for a bipolar fuzzy UP-filter to be a bipolar fuzzy UP-ideal. Further, we discuss the relation between bipolar fuzzy UP-subalgebras (resp., bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals and bipolar fuzzy strongly UP-ideals) and its level cuts.

CHAPTER 2
Basic Results on UP-Algebras

2.1 Basic Results on UP-Algebras

An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* [3] where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

From [3], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 2.1.1. [3] Let X be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of X by putting $A \cdot B = B \cap A'$ and $A * B = B \cup A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, \emptyset)$ and $(\mathcal{P}(X), *, X)$ are UP-algebras and we shall call it the *power UP-algebra of type 1* and the *power UP-algebra of type 2*, respectively.

In what follows, let A be a UP-algebra unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 2.1.2. *[3] In a UP-algebra A, the following properties hold: for any $x, y, z \in A$,*

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

(7) $x \cdot (y \cdot y) = 0$.

Definition 2.1.3. [3] A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.

Iampan [3] proved the useful criteria that a nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.

Definition 2.1.4. [18] A subset F of A is called a *UP-filter* of A if it satisfies the following properties:

(1) the constant 0 of A is in F, and

(2) for any $x, y \in A, x \cdot y \in F$ and $x \in F$ imply $y \in F$.

Definition 2.1.5. [3] A subset B of A is called a *UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in B, and

(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Definition 2.1.6. [1] A subset C of A is called a *strongly UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in C, and

(2) for any $x, y, z \in A, (z \cdot y) \cdot (z \cdot x) \in C$ and $y \in C$ imply $x \in C$.

Theorem 2.1.7. *[3] Let $\{B_i\}_{i \in I}$ be a nonempty family of UP-subalgebras (resp., UP-ideals) of A. Then $\cap_{i \in I} B_i$ is a UP-subalgebra (resp., UP-ideal) of A.*

Theorem 2.1.8. *Let $\{B_i\}_{i\in I}$ be a nonempty family of UP-filters of A. Then $\cap_{i\in I}B_i$ is a UP-filter of A.*

Proof. Since B_i is a UP-filter of A, we have $0 \in B_i$ for all $i \in I$. Thus $0 \in \cap_{i\in I}B_i$. Let $x, y \in A$ be such that $x \cdot y \in \cap_{i\in I}B_i$ and $x \in \cap_{i\in I}B_i$. Then $x \cdot y \in B_i$ and $x \in B_i$ for all $i \in I$. Since B_i is a UP-filter of A, we have $y \in B_i$ for all $i \in I$. Thus $y \in \cap_{i\in I}B_i$. Hence, $\cap_{i\in I}B_i$ is a UP-filter of A. □

Theorem 2.1.9. *Let $\{B_i\}_{i\in I}$ be a nonempty family of strongly UP-ideals of A. Then $\cap_{i\in I}B_i$ is a strongly UP-ideal of A.*

Proof. Since B_i is a strongly UP-ideal of A, we have $0 \in B_i$ for all $i \in I$. Thus $0 \in \bigcap_{i\in I} B_i$. Let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in \bigcap_{i\in I} B_i$ and $y \in \bigcap_{i\in I} B_i$. Then $(z \cdot y) \cdot (z \cdot x) \in B_i$ and $y \in B_i$ for all $i \in I$. Since B_i is a strongly UP-ideal of A, we have $x \in B_i$ for all $i \in I$. Thus $x \in \bigcap_{i\in I} B_i$. Hence, $\bigcap_{i\in I} B_i$ is a strongly UP-ideal of A. □

Guntasow et al. [1] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

CHAPTER 3
Main Results

3.1 Bipolar Fuzzy Sets

Let X be the universe of discourse. A *bipolar-valued fuzzy set* [10] φ in X is an object having the form

$$\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$$

where $\varphi^- : X \to [-1, 0]$ and $\varphi^+ : X \to [0, 1]$ are mappings. For the sake of simplicity, we shall use the symbol $\varphi = (X; \varphi^-, \varphi^+)$ for the bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

Next, we introduce the notion of bipolar fuzzy UP-subalgebras (resp., bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals and bipolar fuzzy strongly UP-ideals) of a UP-algebra A and provide the necessary examples.

Definition 3.1.1. A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy UP-subalgebra* of A if it satisfies the following properties: for any $x, y \in A$,

(1) $\varphi^-(x \cdot y) \leq \max\{\varphi^-(x), \varphi^-(y)\}$, and

(2) $\varphi^+(x \cdot y) \geq \min\{\varphi^+(x), \varphi^+(y)\}$.

Remark 3.1.2. If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A, then

$$\varphi^-(0) \leq \varphi^-(x) \text{ and } \varphi^+(0) \geq \varphi^+(x) \text{ for all } x \in A.$$

Indeed, for all $x \in A$,

$$\varphi^-(0) = \varphi^-(x \cdot x) \leq \max\{\varphi^-(x), \varphi^-(x)\} = \varphi^-(x)$$

and

$$\varphi^+(0) = \varphi^+(x \cdot x) \geq \min\{\varphi^+(x), \varphi^+(x)\} = \varphi^+(x).$$

Example 3.1.3. Consider a UP-algebra $A = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	0	3
2	0	1	0	3
3	0	1	0	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3
φ^-	-0.8	-0.6	-0.2	-0.1
φ^+	0.9	0.7	0.5	0.4

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A.

Definition 3.1.4. A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy UP-filter* of A if it satisfies the following properties: for any $x, y \in A$,

(1) $\varphi^-(0) \leq \varphi^-(x)$,

(2) $\varphi^+(0) \geq \varphi^+(x)$,

(3) $\varphi^-(y) \leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\}$, and

(4) $\varphi^+(y) \geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\}$.

Example 3.1.5. Consider a UP-algebra $A = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3
φ^-	-0.7	-0.3	-0.4	-0.6
φ^+	0.9	0.5	0.1	0.2

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A.

Definition 3.1.6. A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\varphi^-(0) \leq \varphi^-(x)$,

(2) $\varphi^+(0) \geq \varphi^+(x)$,

(3) $\varphi^-(x \cdot z) \leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\}$, and

(4) $\varphi^+(x \cdot z) \geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}$.

Example 3.1.7. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	1	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3	4
φ^-	-0.8	-0.5	-0.5	-0.2	-0.2
φ^+	0.9	0.6	0.6	0.4	0.4

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A.

Definition 3.1.8. A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy strongly UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\varphi^-(0) \leq \varphi^-(x)$,

(2) $\varphi^+(0) \geq \varphi^+(x)$,

(3) $\varphi^-(x) \leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$, and

(4) $\varphi^+(x) \geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}$.

Example 3.1.9. Consider a UP-algebra $A = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	3
3	0	1	0	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3
φ^-	-0.5	-0.5	-0.5	-0.5
φ^+	0.8	0.8	0.8	0.8

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A.

Theorem 3.1.10. *A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is constant if and only if it is a bipolar fuzzy strongly UP-ideal of A.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a constant bipolar fuzzy set in A. Then there exist $l \in [-1, 0]$ and $k \in [0, 1]$ such that

$$\varphi^-(x) = l \text{ for all } x \in A \text{ and } \varphi^+(x) = k \text{ for all } x \in A.$$

Thus $\varphi^-(0) = l \leq l = \varphi^-(x)$ and $\varphi^+(0) = k \geq k = \varphi^+(x)$ for all $x \in A$. For all $x, y, z \in A$,

$$\varphi^-(x) = l \leq l = \max\{l, l\} = \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$$

and

$$\varphi^+(x) = k \geq k = \min\{k, k\} = \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}.$$

Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A.

Conversely, assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A. Then for all $x, y, z \in A$,

$$\varphi^-(0) \leq \varphi^-(x) \text{ and } \varphi^+(0) \geq \varphi^+(x),$$

and

$$\varphi^-(x) \leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\} \text{ and}$$
$$\varphi^+(x) \geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}.$$

For all $x \in A$,

$$\begin{aligned}
\varphi^-(x) &\leq \max\{\varphi^-((x \cdot 0) \cdot (x \cdot x)), \varphi^-(0)\} \\
&\leq \max\{\varphi^-(0 \cdot 0), \varphi^-(0)\} && \text{((UP-3), Proposition 2.1.2 (1))} \\
&\leq \max\{\varphi^-(0), \varphi^-(0)\} && \text{((UP-2))} \\
&= \varphi^-(0).
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+(x) &\geq \min\{\varphi^+((x \cdot 0) \cdot (x \cdot x)), \varphi^+(0)\} \\
&= \min\{\varphi^+(0 \cdot 0), \varphi^+(0)\} && \text{((UP-3), Proposition 2.1.2 (1))} \\
&= \min\{\varphi^+(0), \varphi^+(0)\} && \text{((UP-2))} \\
&= \varphi^+(0).
\end{aligned}$$

Hence, $\varphi^-(x) = \varphi^-(0)$ and $\varphi^+(x) = \varphi^+(0)$ for all $x \in A$. Therefore, $\varphi = (A; \varphi^-, \varphi^+)$ is a constant bipolar fuzzy set in A. □

Theorem 3.1.11. *Every bipolar fuzzy strongly UP-ideal of A is a bipolar fuzzy UP-ideal.*

Proof. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy strongly UP-ideal of A. By Theorem 3.1.10, there exists $(l, k) \in [-1, 0] \times [0, 1]$ such that

$$\varphi^-(x) = l \text{ and } \varphi^+(x) = k \text{ for all } x \in A.$$

For all $x, y, z \in A$,

$$\varphi^-(0) = l \leq l = \varphi^-(x)$$

and

$$\varphi^-(x \cdot z) = l \leq l = \max\{l, l\} = \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\},$$

and

$$\varphi^+(0) = k \geq k = \varphi^+(x)$$

and

$$\varphi^+(x \cdot z) = k \geq k = \min\{k, k\} = \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}.$$

Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A. □

Example 3.1.12. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	1	1	0	4
4	0	1	2	3	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3	4
φ^-	-0.9	-0.6	-0.5	-0.2	-0.7
φ^+	0.8	0.5	0.2	0.1	0.5

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A but it is not a bipolar fuzzy strongly UP-ideal of A. Indeed,

$$\varphi^-(3) = -0.2 > -0.7 = \max\{\varphi^-((3 \cdot 4) \cdot (3 \cdot 3)), \varphi^-(4)\}$$

and

$$\varphi^+(3) = 0.1 < 0.5 = \min\{\varphi^+((3 \cdot 4) \cdot (3 \cdot 3)), \varphi^+(4)\}.$$

Theorem 3.1.13. *Every bipolar fuzzy UP-ideal of A is a bipolar fuzzy UP-filter.*

Proof. Let φ be a bipolar fuzzy UP-ideal of A. Then for all $x, y \in A, \varphi^-(0) \leq \varphi^-(x)$ and

$$\begin{aligned}
\varphi^-(y) &= \varphi^-(0 \cdot y) && \text{((UP-2))} \\
&\leq \max\{\varphi^-(0 \cdot (x \cdot y)), \varphi^-(x)\} \\
&= \max\{\varphi^-(x \cdot y), \varphi^-(x)\}, && \text{((UP-2))}
\end{aligned}$$

and $\varphi^+(0) \geq \varphi^+(x)$ and

$$\begin{aligned}\varphi^+(y) &= \varphi^+(0 \cdot y) && \text{((UP-2))}\\ &\geq \min\{\varphi^+(0 \cdot (x \cdot y)), \varphi^+(x)\}\\ &= \min\{\varphi^+(x \cdot y), \varphi^+(x)\}. && \text{((UP-2))}\end{aligned}$$

Hence, φ is a bipolar fuzzy UP-filter of A. □

Example 3.1.14. Consider a UP-algebra $A = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	3
3	0	0	0	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3
φ^-	-0.7	-0.3	-0.1	-0.1
φ^+	0.8	0.5	0.2	0.2

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A but it is not a bipolar fuzzy UP-ideal of A. Indeed,

$$\varphi^-(2 \cdot 3) = \varphi^-(3) = -0.1 > -0.3 = \max\{\varphi^-(2 \cdot (1 \cdot 3)), \varphi^-(1)\}$$

and

$$\varphi^+(2 \cdot 3) = \varphi^+(3) = 0.2 < 0.5 = \min\{\varphi^+(2 \cdot (1 \cdot 3)), \varphi^+(1)\}.$$

Theorem 3.1.15. *Every bipolar fuzzy UP-filter of A is a bipolar fuzzy UP-subalgebra.*

Proof. Let φ is a bipolar fuzzy UP-filter of A. Then for all $x, y \in A$, $\varphi^-(0) \leq \varphi^-(x)$ and

$$\begin{aligned}\varphi^-(x \cdot y) &\leq \max\{\varphi^-(y \cdot (x \cdot y)), \varphi^-(y)\}\\ &= \max\{\varphi^-(0), \varphi^-(y)\} && \text{(Proposition 2.1.2 (5))}\\ &\leq \max\{\varphi^-(x), \varphi^-(y)\},\end{aligned}$$

and $\varphi^+(0) \geq \varphi^+(x)$ and

$$\begin{aligned}\varphi^+(x \cdot y) &\geq \min\{\varphi^+(y \cdot (x \cdot y)), \varphi^+(y)\} \\ &= \min\{\varphi^+(0), \varphi^+(y)\} && \text{(Proposition 2.1.2 (5))} \\ &\geq \min\{\varphi^+(x), \varphi^+(y)\}.\end{aligned}$$

Hence, φ is a bipolar fuzzy UP-subalgebra of A. □

Example 3.1.16. Consider a UP-algebra $A = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	0	0	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3
φ^+	0.8	0.4	0.2	0.1
φ^-	-0.9	-0.5	-0.3	-0.2

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A but it is not a bipolar fuzzy UP-filter of A. Indeed,

$$\varphi^-(2) = -0.3 > -0.5 = \max\{\varphi^-(1 \cdot 2), \varphi^-(1)\}$$

and

$$\varphi^+(2) = 0.2 < 0.4 = \min\{\varphi^+(1 \cdot 2), \varphi^+(1)\}.$$

3.2 Level Cuts of a Bipolar Fuzzy Set

In this section, we discuss the relation between bipolar fuzzy UP-subalgebras (resp., bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals and bipolar fuzzy strongly UP-ideals) and its level cuts.

Definition 3.2.1. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. For $(t^-, t^+) \in [-1, 0] \times [0, 1]$, the sets

$$N_L(\varphi; t^-) = \{x \in A \mid \varphi^-(x) \leq t^-\}$$

and

$$P_U(\varphi; t^+) = \{x \in A \mid \varphi^+(x) \geq t^+\}$$

are called the *negative lower* t^-*-cut* and the *positive upper* t^+*-cut* of $\varphi = (A; \varphi^-, \varphi^+)$, respectively. The set

$$C(\varphi; (t^-, t^+)) = N_L(\varphi; t^-) \cap P_U(\varphi; t^+)$$

is called the (t^-, t^+)*-cut* of $\varphi = (A; \varphi^-, \varphi^+)$. For any $k \in [0, 1]$, we denote the set

$$C(\varphi; k) = C(\varphi; (-k, k)) = N_L(\varphi; -k) \cap P_U(\varphi; k)$$

is called the k*-cut* of $\varphi = (A; \varphi^-, \varphi^+)$.

Theorem 3.2.2. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A if and only if the following statements are valid:*

(1) for all $t^- \in [-1, 0]$, $N_L(\varphi; t^-)$ is a UP-subalgebra of A if $N_L(\varphi; t^-)$ is nonempty, and

(2) for all $t^+ \in [0, 1]$, $P_U(\varphi; t^+)$ is a UP-subalgebra of A if $P_U(\varphi; t^+)$ is nonempty.

Proof. Assume that φ is a bipolar fuzzy UP-subalgebra of A. Let $t^- \in [-1, 0]$ be such that $N_L(\varphi; t^-) \neq \emptyset$ and let $x, y \in N_L(\varphi; t^-)$. Then $\varphi^-(x) \leq t^-$ and $\varphi^-(y) \leq t^-$. Since φ is a bipolar fuzzy UP-subalgebra of A, we have $\varphi^-(x \cdot y) \leq \max\{\varphi^-(x), \varphi^-(y)\} \leq t^-$. Thus $x \cdot y \in N_L(\varphi; t^-)$. Hence, $N_L(\varphi; t^-)$ is a UP-subalgebra of A. Next, let $t^+ \in [0, 1]$ be such that $P_U(\varphi; t^+) \neq \emptyset$ and let $x, y \in P_U(\varphi; t^+)$. Then $\varphi^+(x) \geq t^+$ and $\varphi^+(y) \geq t^+$. Since φ is a bipolar fuzzy UP-subalgebra of A, we have $\varphi^+(x \cdot y) \geq \min\{\varphi^+(x), \varphi^+(y)\} \geq t^+$. Thus $x \cdot y \in P_U(\varphi; t^+)$. Hence, $P_U(\varphi; t^+)$ is a UP-subalgebra of A.

Conversely, assume that for all $t^- \in [0,1], N_L(\varphi; t^-)$ is a UP-subalgebra of A if $N_L(\varphi; t^-)$ is nonempty and for all $t^+ \in [0,1], P_U(\varphi; t^+)$ is a UP-subalgebra of A if $P_U(\varphi; t^+)$ is nonempty. Let $x, y \in A$. Then $\varphi^-(x), \varphi^-(y) \in [-1,0]$. Choose $t^- = \max\{\varphi^-(x), \varphi^-(y)\}$. Then $\varphi^-(x) \leq t^-$ and $\varphi^-(y) \leq t^-$, that is, $x, y \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a UP-subalgebra of A. So $x \cdot y \in N_L(\varphi; t^-)$. Hence $\varphi^-(x \cdot y) \leq t^- = \max\{\varphi^-(x), \varphi^-(y)\}$. Next, let $x, y \in A$. Then $\varphi^+(x), \varphi^+(y) \in [0,1]$. Choose $t^+ = \min\{\varphi^+(x), \varphi^+(y)\}$. Then $\varphi^+(x) \geq t^+$ and $\varphi^+(y) \geq t^+$, that is, $x, y \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a UP-subalgebra of A. So $x \cdot y \in P_U(\varphi; t^+)$. Hence, $\varphi^+(x \cdot y) \geq t^+ = \min\{\varphi^+(x), \varphi^+(y)\}$. Therefore, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A. □

Corollary 3.2.3. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A, then for all $k \in [0,1], C(\varphi; k)$ is a UP-subalgebra of A while $C(\varphi; k)$ is nonempty.*

Proof. It is straightforward by Theorem 2.1.7 and 3.2.2. □

Theorem 3.2.4. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A if and only if the following statements are valid:*

(1) for all $t^- \in [-1,0], N_L(\varphi; t^-)$ is a UP-filter of A if $N_L(\varphi; t^-)$ is nonempty, and

(2) for all $t^+ \in [0,1], P_U(\varphi; t^+)$ is a UP-filter of A if $P_U(\varphi; t^+)$ is nonempty.

Proof. Assume that φ is a bipolar fuzzy UP-filter of A. Let $t^- \in [-1,0]$ be such that $N_L(\varphi; t^-) \neq \emptyset$ and let $a \in N_L(\varphi; t^-)$. Then $\varphi^-(a) \leq t^-$. Since φ is a bipolar fuzzy UP-filter of A, we have $\varphi^-(0) \leq \varphi^-(a) \leq t^-$. Thus $0 \in N_L(\varphi; t^-)$. Next, let $x, y \in A$ be such that $x \cdot y \in N_L(\varphi; t^-)$ and $x \in N_L(\varphi; t^-)$. Then $\varphi^-(x \cdot y) \leq t^-$ and $\varphi^-(x) \leq t^-$. Since φ is a bipolar fuzzy UP-filter of A, we have

$$\varphi^-(y) \leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\} \leq t^-.$$

So $y \in N_L(\varphi; t^-)$. Hence, $N_L(\varphi; t^-)$ is a UP-filter of A. Let $t^+ \in [0,1]$ be such that $P_U(\varphi; t^+) \neq \emptyset$ and let $a \in P_U(\varphi; t^+)$. Then $\varphi^+(a) \geq t^+$. Since φ is a bipolar fuzzy UP-filter of A, we have $\varphi^+(0) \geq \varphi^+(a) \geq t^+$. Thus $0 \in P_U(\varphi; t^+)$. Next, let $x, y \in A$ be such that $x \cdot y \in P_U(\varphi; t^+)$ and $x \in P_U(\varphi; t^+)$. Then $\varphi^+(x \cdot y) \geq t^+$ and $\varphi^+(x) \geq t^+$. Since φ is a bipolar fuzzy UP-filter of A, we have

$$\varphi^+(y) \geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\} \geq t^+.$$

So $y \in P_U(\varphi; t^+)$. Hence, $P_U(\varphi; t^+)$ is a UP-filter of A.

Conversely, assume that for all $t^- \in [0, 1], N_L(\varphi; t^-)$ is a UP-filter of A if $N_L(\varphi; t^-)$ is nonempty and for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a UP-filter of A if $P_U(\varphi; t^+)$ is nonempty. Let $x \in A$. Then $\varphi^-(x) \in [-1, 0]$. Choose $t^- = \varphi^-(x)$. Then $\varphi^-(x) \leq t^-$, that is, $x \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a UP-filter of A. So $0 \in N_L(\varphi; t^-)$. Hence, $\varphi^-(0) \leq t^- = \varphi^-(x)$. Next, let $x, y \in A$. Then $\varphi^-(x \cdot y), \varphi^-(x) \in [-1, 0]$. Choose $t^- = \max\{\varphi^-(x \cdot y), \varphi^-(x)\}$. Then $\varphi^-(x \cdot y) \leq t^-$ and $\varphi^-(x) \leq t^-$, that is, $x \cdot y, x \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a UP-filter of A. So $y \in N_L(\varphi; t^-)$, Hence, $\varphi^-(y) \leq t^- = \max\{\varphi^-(x), \varphi^-(x \cdot y)\}$. Let $x \in A$. Then $\varphi^+(x) \in [0, 1]$. Choose $t^+ = \varphi^+(x)$. Then $\varphi^+(x) \geq t^+$, that is, $x \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a UP-filter of A. So $0 \in P_U(\varphi; t^+)$. Hence, $\varphi^+(0) \geq t^+ = \varphi^+(x)$. Next, let $x, y \in A$. Then $\varphi^+(x \cdot y), \varphi^+(x) \in [0, 1]$. Choose $t^+ = \min\{\varphi^+(x \cdot y), \varphi^+(x)\}$. Then $\varphi^+(x \cdot y) \geq t^+$ and $\varphi^+(x) \geq t^+$, that is, $x \cdot y, x \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a UP-filter of A. So $y \in P_U(\varphi; t^+)$. Hence, $\varphi^+(y) \geq t^+ = \min\{\varphi^+(x), \varphi^+(x \cdot y)\}$. Therefore, φ is a bipolar fuzzy UP-filter of A. □

Corollary 3.2.5. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A,then for all $k \in [0, 1], C(\varphi; k)$ is a UP-filter of A while $C(\varphi; k)$ is nonempty.*

Proof. It is straightforward by Theorem 2.1.8 and 3.2.4. □

Theorem 3.2.6. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A if and only if the following statements are valid:*

(1) for all $t^- \in [-1, 0], N_L(\varphi; t^-)$ is a UP-ideal of A if $N_L(\varphi; t^-)$ is nonempty, and

(2) for all $t^+ \in [0,1], P_U(\varphi; t^+)$ *is a UP-ideal of* A *if* $P_U(\varphi; t^+)$ *is nonempty.*

Proof. Assume that φ is a bipolar fuzzy UP-ideal of A. Let $t^- \in [-1,0]$ be such that $N_L(\varphi; t^-) \neq \emptyset$ and let $a \in N_L(\varphi; t^-)$. Then $\varphi^-(a) \leq t^-$. Since φ is a bipolar fuzzy UP-ideal of A, we have $\varphi^-(0) \leq \varphi^-(a) \leq t^-$. Thus $0 \in N_L(\varphi; t^-)$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in N_L(\varphi; t^-)$ and $y \in N_L(\varphi; t^-)$. Then $\varphi^-(x \cdot (y \cdot z)) \leq t^-$ and $\varphi^-(y) \leq t^-$. Since φ is a bipolar fuzzy UP-ideal of A, we have

$$\varphi^-(x \cdot z) \leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\} \leq t^-.$$

So $x \cdot z \in N_L(\varphi; t^-)$. Hence, $N_L(\varphi; t^-)$ is a UP-ideal of A. Let $t^+ \in [0,1]$ be such that $P_U(\varphi; t^+) \neq \emptyset$ and let $a \in P_U(\varphi; t^+)$. Then $\varphi^+(a) \geq t^+$. Since φ is a bipolar fuzzy UP-ideal of A, we have $\varphi^+(0) \geq \varphi^+(a) \geq t^+$. Thus $0 \in P_U(\varphi; t^+)$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in P_U(\varphi; t^+)$ and $y \in P_U(\varphi; t^+)$. Then $\varphi^+(x \cdot (y \cdot z)) \geq t^+$ and $\varphi^+(y) \geq t^+$. Since φ is a bipolar fuzzy UP-ideal of A, we have

$$\varphi^+(x \cdot z) \geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\} \geq t^+.$$

So $x \cdot z \in P_U(\varphi; t^+)$. Hence, $P_U(\varphi; t^+)$ is a UP-ideal of A.

Conversely, assume that for all $t^- \in [0,1], N_L(\varphi; t^-)$ is a UP-ideal of A if $N_L(\varphi; t^-)$ is nonempty and for all $t^+ \in [0,1], P_U(\varphi; t^+)$ is a UP-ideal of A if $P_U(\varphi; t^+)$ is nonempty. Let $x \in A$. Then $\varphi^-(x) \in [-1,0]$. Choose $t^- = \varphi^-(x)$. Then $\varphi^-(x) \leq t^-$, that is, $x \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a UP-ideal of A. So $0 \in N_L(\varphi; t^-)$. Hence, $\varphi^-(0) \leq t^- = \varphi^-(x)$. Next, let $x, y, z \in A$. Then $\varphi^-(x \cdot (y \cdot z)), \varphi^-(y) \in [-1,0]$. Choose $t^- = \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\}$. Then $\varphi^-(x \cdot (y \cdot z)) \leq t^-$ and $\varphi^-(y) \leq t^-$, that is, $x \cdot (y \cdot z), y \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a UP-ideal of A. So $x \cdot z \in N_L(\varphi; t^-)$. Hence, $\varphi^-(x \cdot z) \leq t^- = \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\}$. Let $x \in A$. Then $\varphi^+(x) \in [0,1]$. Choose $t^+ = \varphi^+(x)$. Then $\varphi^+(x) \geq t^+$, that is, $x \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a UP-ideal of A. So $0 \in P_U(\varphi; t^+)$. Hence, $\varphi^+(0) \geq t^+ = \varphi^+(x)$. Next, let $x, y, z \in A$. Then $\varphi^+(x \cdot (y \cdot z)), \varphi^+(y) \in [0,1]$. Choose $t^+ = \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}$. Then $\varphi^+(x \cdot (y \cdot z)) \geq t^+$ and $\varphi^+(y) \geq t^+$, that is, $x \cdot (y \cdot z), y \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a UP-ideal of A. So $x \cdot z \in P_U(\varphi; t^+)$. Hence, $\varphi^+(x \cdot z) \geq t^+ = \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}$. Therefore, φ is a bipolar fuzzy UP-ideal of A. □

Corollary 3.2.7. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A, then for all $k \in [0,1], C(\varphi; k)$ is a UP-ideal of A while $C(\varphi; k)$ is nonempty.*

Proof. It is straightforward by Theorem 2.1.7 and 3.2.6. □

Give an example of conflict that the converse of Corollary 3.2.5, , and 3.2.7 is not true.

Example 3.2.8. Consider a UP-algebra $A = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3
φ^-	-0.6	-0.6	-0.3	-0.6
φ^+	0.6	0.3	0.6	0.3

Then for all $k \in [0,1], C(\varphi; k)$ is a UP-subalgebra (resp., UP-filter, UP-ideal) of A while $C(\varphi; k)$ is nonempty. Indeed,

(1) $C(\varphi; k) = A$ if $k \in [0, 0.3]$,

(2) $C(\varphi; k) = \{0\}$ if $k \in (0.3, 0.6]$, and

(3) $C(\varphi; k) = \emptyset$ if $k \in (0.6, 1]$.

But φ is not a bipolar fuzzy UP-subalgebra of A. Indeed,

$$\varphi^-(1 \cdot 3) = -0.3 > -0.6 = \max\{-0.6, -0.6\} = \max\{\varphi^-(1), \varphi^-(3)\}.$$

By Theorem 3.1.15 and Theorem 3.1.13, we have φ is not a bipolar fuzzy UP-filter and a bipolar fuzzy UP-ideal of A.

Theorem 3.2.9. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A if and only if the following statements are valid:*

(1) for all $t^- \in [-1, 0], N_L(\varphi; t^-)$ is a strongly UP-ideal of A if $N_L(\varphi; t^-)$ is nonempty, and

(2) for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a strongly UP-ideal of A if $P_U(\varphi; t^+)$ is nonempty.

Proof. Assume that φ is a bipolar fuzzy strongly UP-ideal of A. Let $t^- \in [-1, 0]$ be such that $N_L(\varphi; t^-) \neq \emptyset$ and let $a \in N_L(\varphi; t^-)$. Then $\varphi^-(a) \leq t^-$. Since φ is a bipolar fuzzy strongly UP-ideal of A, we have $\varphi^-(0) \leq \varphi^-(a) \leq t^-$. Thus $0 \in N_L(\varphi; t^-)$. Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in N_L(\varphi; t^-)$ and $y \in N_L(\varphi; t^-)$. Then $\varphi^-((z \cdot y) \cdot (z \cdot x)) \leq t^-$ and $\varphi^-(y) \leq t^-$. Since φ is a bipolar fuzzy strongly UP-ideal of A, we have

$$\varphi^-(x) \leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\} \leq t^-.$$

So $x \in N_L(\varphi; t^-)$. Hence, $N_L(\varphi; t^-)$ is a strongly UP-ideal of A. Let $t^+ \in [0, 1]$ be such that $P_U(\varphi; t^+) \neq \emptyset$ and let $a \in P_U(\varphi; t^+)$. Then $\varphi^+(a) \geq t^+$. Since φ is a bipolar fuzzy strongly UP-ideal of A, we have $\varphi^+(0) \geq \varphi^+(a) \geq t^+$. Thus $0 \in P_U(\varphi; t^+)$. Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in P_U(\varphi; t^+)$ and $y \in P_U(\varphi; t^+)$. Then $\varphi^+((z \cdot y) \cdot (z \cdot x)) \geq t^+$ and $\varphi^+(y) \geq t^+$. Since φ is a bipolar fuzzy strongly UP-ideal of A, we have

$$\varphi^+(x) \geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\} \geq t^+.$$

So $x \in P_U(\varphi; t^+)$. Hence, $P_U(\varphi; t^+)$ is a strongly UP-ideal of A.

Conversely, assume that for all $t^- \in [0, 1], N_L(\varphi; t^-)$ is a strongly UP-ideal of A if $N_L(\varphi; t^-)$ is nonempty and for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a strongly UP-ideal of A if $P_U(\varphi; t^+)$ is nonempty. Let $x \in A$. Then $\varphi^-(x) \in [-1, 0]$. Choose $t^- = \varphi^-(x)$. Then $\varphi^-(x) \leq t^-$, that is, $x \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a strongly UP-ideal of A. So $0 \in N_L(\varphi; t^-)$. Hence, $\varphi^-(0) \leq t^- = \varphi^-(x)$. Next, let $x, y, z \in A$. Then $\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y) \in [-1, 0]$. Choose $t^- = \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$. Then $\varphi^-((z \cdot y) \cdot (z \cdot x)) \leq t^-$ and $\varphi^-(y) \leq t^-$, that is, $(z \cdot y) \cdot (z \cdot x), y \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a strongly UP-ideal of A. So $x \in N_L(\varphi; t^-)$. Hence, $\varphi^-(x) \leq t^- = \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$. Let $x \in A$. Then $\varphi^+(x) \in [0, 1]$. Choose $t^+ = \varphi^+(x)$. Then $\varphi^+(x) \geq t^+$, that is, $x \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a strongly UP-ideal of A. So $0 \in P_U(\varphi; t^+)$. Hence, $\varphi^+(0) \geq t^+ = \varphi^+(x)$. Next, let $x, y, z \in A$. Then $\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y) \in [0, 1]$. Choose $t^+ = \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}$. Then $\varphi^+((z \cdot y) \cdot (z \cdot x)) \geq t^+$ and $\varphi^+(y) \geq t^+$, that is, $(z \cdot y) \cdot (z \cdot x), y \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a strongly UP-ideal of A. So $x \in P_U(\varphi; t^+)$. Hence, $\varphi^+(x) \geq t^+ = \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}$. Therefore, φ is a bipolar fuzzy strongly UP-ideal of A. □

Corollary 3.2.10. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A if and only if for all $k \in [0, 1], C(\varphi; k)$ is a strongly UP-ideal of A while $C(\varphi; k)$ is nonempty.*

Proof. It is straightforward by Theorem 2.1.9, 3.2.9, and 3.1.10, and A is the only one strongly UP-ideal of itself. □

Theorem 3.2.11. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-filter of A satisfies the following assertion:*

$$x \cdot (y \cdot z) = y \cdot (x \cdot z) \text{ for all } x, y, z \in A. \tag{3.1}$$

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A.

Proof. For all $x, y, z \in A$,

$$\varphi^-(0) \leq \varphi^-(x)$$

and

$$\begin{aligned} \varphi^-(x \cdot z) &\leq \max\{\varphi^-(y \cdot (x \cdot z)), \varphi^-(y)\} \\ &= \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\}, \end{aligned} \tag{(4.2)}$$

and

$$\varphi^+(0) \geq \varphi^+(x)$$

and

$$\begin{aligned}\varphi^+(x \cdot z) &\geq \min\{\varphi^+(y \cdot (x \cdot z)), \varphi^+(y)\}\\ &= \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}. \qquad ((4.2))\end{aligned}$$

Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A. □

Definition 3.2.12. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. We define a subset $\varphi^{-1}(0,0)$ of A by

$$\varphi^{-1}(0,0) = \{x \in A \mid \varphi^-(x) = \varphi^-(0) \text{ and } \varphi^+(x) = \varphi^+(0)\}.$$

Theorem 3.2.13. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-subalgebra of A. Then $\varphi^{-1}(0,0)$ is a UP-subalgebra of A.*

Proof. Clearly, $0 \in \varphi^{-1}(0,0)$. Let $x, y \in \varphi^{-1}(0,0)$. Then $\varphi^-(x) = \varphi^-(0), \varphi^+(x) = \varphi^+(0), \varphi^-(y) = \varphi^-(0)$, and $\varphi^+(y) = \varphi^+(0)$. Thus

$$\begin{aligned}\varphi^-(0) &\leq \varphi^-(x \cdot y)\\ &\leq \max\{\varphi^-(x), \varphi^-(y)\}\\ &= \max\{\varphi^-(0), \varphi^-(0)\}\\ &= \varphi^-(0)\end{aligned}$$

and

$$\begin{aligned}\varphi^+(0) &\geq \varphi^+(x \cdot y)\\ &\geq \min\{\varphi^+(x), \varphi^+(y)\}\\ &= \min\{\varphi^+(0), \varphi^+(0)\}\\ &= \varphi^+(0).\end{aligned}$$

So $\varphi^-(x \cdot y) = \varphi^-(0)$ and $\varphi^+(x \cdot y) = \varphi^+(0)$, that is, $x \cdot y \in \varphi^{-1}(0,0)$. Therefore, $\varphi^{-1}(0,0)$ is a UP-subalgebra of A. □

Theorem 3.2.14. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-filter of A. Then $\varphi^{-1}(0,0)$ is a UP-filter of A.*

Proof. Clearly, $0 \in \varphi^{-1}(0,0)$. Let $x, y \in A$ be such that $x \cdot y \in \varphi^{-1}(0,0)$ and $x \in \varphi^{-1}(0,0)$. Then $\varphi^-(x) = \varphi^-(0), \varphi^+(x) = \varphi^+(0), \varphi^-(x \cdot y) = \varphi^-(0)$, and $\varphi^+(x \cdot y) = \varphi^+(0)$. Thus

$$\begin{aligned}\varphi^-(0) &\leq \varphi^-(y) \\ &\leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\} \\ &= \max\{\varphi^-(0), \varphi^-(0)\} \\ &= \varphi^-(0)\end{aligned}$$

and

$$\begin{aligned}\varphi^+(0) &\geq \varphi^+(y) \\ &\geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\} \\ &= \min\{\varphi^+(0), \varphi^+(0)\} \\ &= \varphi^+(0).\end{aligned}$$

So $\varphi^-(y) = \varphi^-(0)$ and $\varphi^+(y) = \varphi^+(0)$, that is, $y \in \varphi^{-1}(0,0)$. Therefore, $\varphi^{-1}(0,0)$ is a UP-filter of A. □

Theorem 3.2.15. *Let* $\varphi = (A; \varphi^-, \varphi^+)$ *be a bipolar fuzzy UP-ideal of* A. *Then* $\varphi^{-1}(0,0)$ *is a UP-ideal of* A.

Proof. Clearly, $0 \in \varphi^{-1}(0,0)$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in \varphi^{-1}(0,0)$ and $y \in \varphi^{-1}(0,0)$. Then $\varphi^-(x \cdot (y \cdot z)) = \varphi^-(0), \varphi^+(x \cdot (y \cdot z)) = \varphi^+(0), \varphi^-(y) = \varphi^-(0)$, and $\varphi^+(y) = \varphi^+(0)$. Thus

$$\begin{aligned}\varphi^-(0) &\leq \varphi^-(x \cdot z) \\ &\leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\} \\ &= \max\{\varphi^-(0), \varphi^-(0)\} \\ &= \varphi^-(0)\end{aligned}$$

and

$$\begin{aligned}\varphi^+(0) &\geq \varphi^+(x \cdot z)\\ &\geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}\\ &= \min\{\varphi^+(0), \varphi^+(0)\}\\ &= \varphi^+(0).\end{aligned}$$

So $\varphi^-(x \cdot z) = \varphi^-(0)$ and $\varphi^+(x \cdot z) = \varphi^+(0)$, that is, $x \cdot z \in \varphi^{-1}(0, 0)$. Therefore, $\varphi^{-1}(0, 0)$ is a UP-ideal of A. □

Give an example of conflict that the converse of Theorem 3.2.13, 3.2.14, and 3.2.15 is not true.

Example 3.2.16. From Example 3.2.8, we have $\varphi^{-1}(0, 0) = \{0\}$ is a UP-subalgebra (resp., UP-filter, UP-ideal) of A but φ is not a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-ideal, bipolar fuzzy UP-filter) of A.

Theorem 3.2.17. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A if and only if $\varphi^{-1}(0, 0)$ is a strongly UP-ideal of A.*

Proof. It is straightforward by Theorem 3.1.10, and A is the only one strongly UP-ideal of itself. □

CHAPTER 4
Conclusions

From the study, we get the main results as the following:

1. A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is constant if and only if it is a bipolar fuzzy strongly UP-ideal of A.

2. Every bipolar fuzzy strongly UP-ideal of A is a bipolar fuzzy UP-ideal.

3. Every bipolar fuzzy UP-ideal of A is a bipolar fuzzy UP-filter.

4. Every bipolar fuzzy UP-filter of A is a bipolar fuzzy UP-subalgebra.

5. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A if and only if the following statements are valid:

   (1) for all $t^- \in [-1, 0], N_L(\varphi; t^-)$ is a UP-subalgebra of A if $N_L(\varphi; t^-)$ is nonempty, and

   (2) for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a UP-subalgebra of A if $P_U(\varphi; t^+)$ is nonempty.

6. If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A, then for all $k \in [0, 1], C(\varphi; k)$ is a UP-subalgebra of A while $C(\varphi; k)$ is nonempty.

7. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A if and only if the following statements are valid:

   (1) for all $t^- \in [-1, 0], N_L(\varphi; t^-)$ is a UP-filter of A if $N_L(\varphi; t^-)$ is nonempty, and

   (2) for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a UP-filter of A if $P_U(\varphi; t^+)$ is nonempty.

8. If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A,then for all $k \in [0, 1], C(\varphi; k)$ is a UP-filter of A while $C(\varphi; k)$ is nonempty.

9. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A if and only if the following statements are valid:

   (1) for all $t^- \in [-1, 0], N_L(\varphi; t^-)$ is a UP-ideal of A if $N_L(\varphi; t^-)$ is nonempty, and

   (2) for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a UP-ideal of A if $P_U(\varphi; t^+)$ is nonempty.

10. If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A, then for all $k \in [0, 1], C(\varphi; k)$ is a UP-ideal of A while $C(\varphi; k)$ is nonempty.

11. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A if and only if the following statements are valid:

   (1) for all $t^- \in [-1, 0], N_L(\varphi; t^-)$ is a strongly UP-ideal of A if $N_L(\varphi; t^-)$ is nonempty, and

   (2) for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a strongly UP-ideal of A if $P_U(\varphi; t^+)$ is nonempty.

12. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A if and only if for all $k \in [0, 1], C(\varphi; k)$ is a strongly UP-ideal of A while $C(\varphi; k)$ is nonempty.

13. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-filter of A satisfies the following assertion:

$$x \cdot (y \cdot z) = y \cdot (x \cdot z) \ for \ all \ x, y, z \in A. \tag{4.2}$$

   Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A.

14. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-subalgebra of A. Then $\varphi^{-1}(0, 0)$ is a UP-subalgebra of A.

15. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-filter of A. Then $\varphi^{-1}(0, 0)$ is a UP-filter of A.

16. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-ideal of A. Then $\varphi^{-1}(0,0)$ is a UP-ideal of A.

17. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A if and only if $\varphi^{-1}(0,0)$ is a strongly UP-ideal of A.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] T. Guntasow, S. Sajak, A. Jomkham, and A. Iampan, Fuzzy translations of a fuzzy set in UP-algebras, J. Indones. Math. Soc. **23** (2017), no. 2, 1–19.

[2] Q. P. Hu and X. Li, On BCH-algebras, Math. Semin. Notes, Kobe Univ. **11** (1983), 313–320.

[3] A. Iampan, A new branch of the logical algebra: UP-algebras, J. Algebra Relat. Top. **5** (2017), no. 1, 35–54.

[4] Y. Imai and K. Iséki, On axiom system of propositional calculi, XIV, Proc. Japan Acad. **42** (1966), no. 1, 19–22.

[5] K. Iséki, An algebra related with a propositional calculus, Proc. Japan Acad. **42** (1966), no. 1, 26–29.

[6] Y. B. Jun, K. J. Lee, and E. H. Roh, Ideals and filters in CI-algebras based on bipolar-valued fuzzy sets, Ann. Fuzzy Math. Inform. **4** (2012), no. 1, 109–121.

[7] Y. B. Jun and C. H. Park, Filters of BCH-algebras based on bipolar-valued fuzzy sets, Int. Math. Forum **4** (2009), no. 13, 631–643.

[8] Y. B. Jun and S. Z. Song, Subalgebras and closed ideals of BCH-algebras based on bipolar-valued fuzzy sets, Sci. Math. Jpn **68** (2008), no. 2, 287–297.

[9] S. Keawrahun and U. Leerawat, On isomorphisms of SU-algebras, Sci. Magna **7** (2011), no. 2, 39–44.

[10] K. J. Lee and Y. B. Jun, Bipolar fuzzy a-ideals of BCI-algebras, Commun. Korean Math. Soc. **26** (2011), no. 4, 531–542.

[11] K. M. Lee, Bipolar-valued fuzzy sets and their operations, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand (2000), 307–312.

[12] G. Muhiuddin, Bipolar fuzzy KU-subalgebras/ideals of KU-algebras, Ann. Fuzzy Math. Inform. **8** (2014), no. 3, 409–418.

[13] C. Prabpayak and U. Leerawat, On ideals and congruences in KU-algebras, Sci. Magna **5** (2009), no. 1, 54–57.

[14] S. Sabarinathan, D. C. Kumar, and P. Muralikrishna, Bipolar valued fuzzy α-ideal of BF-algebra, Circuits Syst. **7** (2016), 3054–3062.

[15] S. Sabarinathan, D. C. Kumar, and P. Muralikrishna, Bipolar valued fuzzy ideals of BF-algebras, Int. J. Pure Appl. Math. **109** (2016), no. 4, 837–846.

[16] S. Sabarinathan, P. Muralikrishna, and D. C. Kumar, Bipolar valued fuzzy H-ideals of BF-algebras, Int. J. Pure Appl. Math. **112** (2017), no. 5, 87–92.

[17] T. Senapati, Bipolar fuzzy structure of BG-subalgebras, J. Fuzzy Math. **23** (2015), no. 1, 209–220.

[18] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, and A. Iampan, Fuzzy sets in UP-algebras, Ann. Fuzzy Math. Inform. **12** (2016), no. 6, 739–756.

[19] L. A. Zadeh, Fuzzy sets, Inf. Cont. **8** (1965), 338–353.

APPENDIX

Writing for Publication: Bipolar Fuzzy UP-Algebras*

Korawut Kawila (korawutkawila@gmail.com)
Chaiphon Udomsetchai (lovesumen3@gmail.com)
Aiyared Iampan†(aiyared.ia@up.ac.th)

Department of Mathematics, School of Science
University of Phayao, Phayao 56000, Thailand

Friday 27th April, 2018

Abstract

In this paper, we apply the notion of bipolar-valued fuzzy set to UP-algebras. We introduce the notions of bipolar fuzzy UP-subalgebras (resp., bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals and bipolar fuzzy strongly UP-ideals) of UP-algebras and prove its generalizations. We provide a condition for a bipolar fuzzy UP-filter to be a bipolar fuzzy UP-ideal. Further, we discuss the relation between bipolar fuzzy UP-subalgebras (resp., bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals and bipolar fuzzy strongly UP-ideals) and its level cuts.

Mathematics Subject Classification: 03G25, 08A72
Keywords: UP-algebra, bipolar fuzzy UP-subalgebra, bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal and bipolar fuzzy strongly UP-ideal

1 Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [4], BCI-algebras [5], BCH-algebras [2], KU-algebras [13], SU-algebras [9], UP-algebras [3] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [5] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [4, 5] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

The notion of fuzzy subsets of a set was first considered by Zadeh [19] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. There are several kinds of fuzzy set extensions in the fuzzy set

*This work was financially supported by the University of Phayao.
†Corresponding author.

theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, bipolar-valued fuzzy sets etc. The notion of bipolar-valued fuzzy sets was first introduced by Lee [11] in 2000, is an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 0]$.

After the introduction of the notion of bipolar-valued fuzzy sets by Lee [11], several researches were conducted on the generalizations of the notion of bipolar-valued fuzzy sets and application to many logical algebras such as: In 2008, Jun and Song [8] introduced the notions of bipolar fuzzy subalgebras and bipolar fuzzy closed ideals in BCH-algebras. In 2009, Jun and Park [7] introduced the notions of bipolar fuzzy regularities, bipolar fuzzy regular subalgebras, bipolar fuzzy filters, and bipolar fuzzy closed quasi filters in BCH-algebras. In 2011, Lee and Jun [10] introduced the notion of bipolar fuzzy a-ideals of BCI-algebras. In 2012, Jun et al. [6] introduced the notions of bipolar fuzzy CI-subalgebras, bipolar fuzzy ideals and (closed) bipolar fuzzy filters in CI-algebras. In 2014, Muhiuddin [12] introduced the notions of bipolar fuzzy KU-subalgebras and bipolar fuzzy KU-ideals in KU-algebras. In 2015, Senapati [17] introduced the notion of bipolar fuzzy BG-subalgebras in BG-algebras. In 2016, Sabarinathan et al. [15] introduced the notion of bipolar valued fuzzy ideals of BF-algebras. Sabarinathan et al. [14] introduced the notion of bipolar valued fuzzy α-ideals of BF-algebras. In 2017, Sabarinathan et al. [16] introduced the notion of bipolar valued fuzzy H-ideals of BF-algebras.

In this paper, we apply the notion of bipolar-valued fuzzy set to UP-algebras. We introduce the notions of bipolar fuzzy UP-subalgebras (resp., bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals and bipolar fuzzy strongly UP-ideals) of UP-algebras and prove its generalizations. We provide a condition for a bipolar fuzzy UP-filter to be a bipolar fuzzy UP-ideal. Further, we discuss the relation between bipolar fuzzy UP-subalgebras (resp., bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals and bipolar fuzzy strongly UP-ideals) and its level cuts.

2 Basic Results on UP-Algebras

An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* [3] where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

From [3], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 2.1. [3] Let X be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of X by putting $A \cdot B = B \cap A'$ and $A * B = B \cup A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, \emptyset)$ and $(\mathcal{P}(X), *, X)$ are UP-algebras and we shall call it the *power UP-algebra of type 1* and the *power UP-algebra of type 2*, respectively.

In what follows, let A be a UP-algebra unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 2.2. *[3] In a UP-algebra A, the following properties hold: for any $x, y, z \in A$,*

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

(7) $x \cdot (y \cdot y) = 0$.

Definition 2.3. [3] A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.

Iampan [3] proved the useful criteria that a nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.

Definition 2.4. [18] A subset F of A is called a *UP-filter* of A if it satisfies the following properties:

(1) the constant 0 of A is in F, and

(2) for any $x, y \in A, x \cdot y \in F$ and $x \in F$ imply $y \in F$.

Definition 2.5. [3] A subset B of A is called a *UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in B, and

(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Definition 2.6. [1] A subset C of A is called a *strongly UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in C, and

(2) for any $x, y, z \in A, (z \cdot y) \cdot (z \cdot x) \in C$ and $y \in C$ imply $x \in C$.

Theorem 2.7. *[3] Let* $\{B_i\}_{i \in I}$ *be a nonempty family of UP-subalgebras (resp., UP-ideals) of* A. *Then* $\cap_{i \in I} B_i$ *is a UP-subalgebra (resp., UP-ideal) of* A.

Theorem 2.8. *Let* $\{B_i\}_{i \in I}$ *be a nonempty family of UP-filters of* A. *Then* $\cap_{i \in I} B_i$ *is a UP-filter of* A.

Proof. Since B_i is a UP-filter of A, we have $0 \in B_i$ for all $i \in I$. Thus $0 \in \cap_{i \in I} B_i$. Let $x, y \in A$ be such that $x \cdot y \in \cap_{i \in I} B_i$ and $x \in \cap_{i \in I} B_i$. Then $x \cdot y \in B_i$ and $x \in B_i$ for all $i \in I$. Since B_i is a UP-filter of A, we have $y \in B_i$ for all $i \in I$. Thus $y \in \cap_{i \in I} B_i$. Hence, $\cap_{i \in I} B_i$ is a UP-filter of A. □

Theorem 2.9. *Let* $\{B_i\}_{i \in I}$ *be a nonempty family of strongly UP-ideals of* A. *Then* $\cap_{i \in I} B_i$ *is a strongly UP-ideal of* A.

Proof. Since B_i is a strongly UP-ideal of A, we have $0 \in B_i$ for all $i \in I$. Thus $0 \in \bigcap_{i \in I} B_i$. Let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in \bigcap_{i \in I} B_i$ and $y \in \bigcap_{i \in I} B_i$. Then $(z \cdot y) \cdot (z \cdot x) \in B_i$ and $y \in B_i$ for all $i \in I$. Since B_i is a strongly UP-ideal of A, we have $x \in B_i$ for all $i \in I$. Thus $x \in \bigcap_{i \in I} B_i$. Hence, $\bigcap_{i \in I} B_i$ is a strongly UP-ideal of A. □

Guntasow et al. [1] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

3 Bipolar Fuzzy Sets

Let X be the universe of discourse. A *bipolar-valued fuzzy set* [11] φ in X is an object having the form

$$\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$$

where $\varphi^- : X \to [-1, 0]$ and $\varphi^+ : X \to [0, 1]$ are mappings. For the sake of simplicity, we shall use the symbol $\varphi = (X; \varphi^-, \varphi^+)$ for the bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

Next, we introduce the notion of bipolar fuzzy UP-subalgebras (resp., bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals and bipolar fuzzy strongly UP-ideals) of a UP-algebra A and provide the necessary examples.

Definition 3.1. A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy UP-subalgebra* of A if it satisfies the following properties: for any $x, y \in A$,

(1) $\varphi^-(x \cdot y) \leq \max\{\varphi^-(x), \varphi^-(y)\}$, and

(2) $\varphi^+(x \cdot y) \geq \min\{\varphi^+(x), \varphi^+(y)\}$.

Remark 3.2. If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A, then

$$\varphi^-(0) \leq \varphi^-(x) \text{ and } \varphi^+(0) \geq \varphi^+(x) \text{ for all } x \in A.$$

Indeed, for all $x \in A$,

$$\varphi^-(0) = \varphi^-(x \cdot x) \leq \max\{\varphi^-(x), \varphi^-(x)\} = \varphi^-(x)$$

and

$$\varphi^+(0) = \varphi^+(x \cdot x) \geq \min\{\varphi^+(x), \varphi^+(x)\} = \varphi^+(x).$$

Example 3.3. Consider a UP-algebra $A = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	0	3
2	0	1	0	3
3	0	1	0	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3
φ^-	-0.8	-0.6	-0.2	-0.1
φ^+	0.9	0.7	0.5	0.4

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A.

Definition 3.4. A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy UP-filter* of A if it satisfies the following properties: for any $x, y \in A$,

(1) $\varphi^-(0) \leq \varphi^-(x)$,

(2) $\varphi^+(0) \geq \varphi^+(x)$,

(3) $\varphi^-(y) \leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\}$, and

(4) $\varphi^+(y) \geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\}$.

Example 3.5. Consider a UP-algebra $A = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3
φ^-	-0.7	-0.3	-0.4	-0.6
φ^+	0.9	0.5	0.1	0.2

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A.

Definition 3.6. A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\varphi^-(0) \leq \varphi^-(x)$,

(2) $\varphi^+(0) \geq \varphi^+(x)$,

(3) $\varphi^-(x \cdot z) \leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\}$, and

(4) $\varphi^+(x \cdot z) \geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}$.

Example 3.7. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	1	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3	4
φ^-	-0.8	-0.5	-0.5	-0.2	-0.2
φ^+	0.9	0.6	0.6	0.4	0.4

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A.

Definition 3.8. A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy strongly UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\varphi^-(0) \leq \varphi^-(x)$,

(2) $\varphi^+(0) \geq \varphi^+(x)$,

(3) $\varphi^-(x) \leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$, and

(4) $\varphi^+(x) \geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}$.

Example 3.9. Consider a UP-algebra $A = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	3
3	0	1	0	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3
φ^-	-0.5	-0.5	-0.5	-0.5
φ^+	0.8	0.8	0.8	0.8

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A.

Theorem 3.10. *A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is constant if and only if it is a bipolar fuzzy strongly UP-ideal of A.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a constant bipolar fuzzy set in A. Then there exist $l \in [-1, 0]$ and $k \in [0, 1]$ such that

$$\varphi^-(x) = l \text{ for all } x \in A \text{ and } \varphi^+(x) = k \text{ for all } x \in A.$$

Thus $\varphi^-(0) = l \leq l = \varphi^-(x)$ and $\varphi^+(0) = k \geq k = \varphi^+(x)$ for all $x \in A$. For all $x, y, z \in A$,

$$\varphi^-(x) = l \leq l = \max\{l, l\} = \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$$

and

$$\varphi^+(x) = k \geq k = \min\{k, k\} = \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}.$$

Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A.

Conversely, assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A. Then for all $x, y, z \in A$,

$$\varphi^-(0) \leq \varphi^-(x) \text{ and } \varphi^+(0) \geq \varphi^+(x),$$

and

$$\varphi^-(x) \leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\} \text{ and } \varphi^+(x) \geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}.$$

For all $x \in A$,

$$\begin{aligned}
\varphi^-(x) &\leq \max\{\varphi^-((x \cdot 0) \cdot (x \cdot x)), \varphi^-(0)\} \\
&\leq \max\{\varphi^-(0 \cdot 0), \varphi^-(0)\} && \text{((UP-3), Proposition 2.2 (1))} \\
&\leq \max\{\varphi^-(0), \varphi^-(0)\} && \text{((UP-2))} \\
&= \varphi^-(0).
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+(x) &\geq \min\{\varphi^+((x \cdot 0) \cdot (x \cdot x)), \varphi^+(0)\} \\
&= \min\{\varphi^+(0 \cdot 0), \varphi^+(0)\} && \text{((UP-3), Proposition 2.2 (1))} \\
&= \min\{\varphi^+(0), \varphi^+(0)\} && \text{((UP-2))} \\
&= \varphi^+(0).
\end{aligned}$$

Hence, $\varphi^-(x) = \varphi^-(0)$ and $\varphi^+(x) = \varphi^+(0)$ for all $x \in A$. Therefore, $\varphi = (A; \varphi^-, \varphi^+)$ is a constant bipolar fuzzy set in A. □

Theorem 3.11. *Every bipolar fuzzy strongly UP-ideal of A is a bipolar fuzzy UP-ideal.*

Proof. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy strongly UP-ideal of A. By Theorem 3.10, there exists $(l, k) \in [-1, 0] \times [0, 1]$ such that

$$\varphi^-(x) = l \text{ and } \varphi^+(x) = k \text{ for all } x \in A.$$

For all $x, y, z \in A$,

$$\varphi^-(0) = l \leq l = \varphi^-(x)$$

and

$$\varphi^-(x \cdot z) = l \leq l = \max\{l, l\} = \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\},$$

and

$$\varphi^+(0) = k \geq k = \varphi^+(x)$$

and

$$\varphi^+(x \cdot z) = k \geq k = \min\{k, k\} = \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}.$$

Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A. □

Example 3.12. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	1	1	0	4
4	0	1	2	3	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3	4
φ^-	-0.9	-0.6	-0.5	-0.2	-0.7
φ^+	0.8	0.5	0.2	0.1	0.5

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A but it is not a bipolar fuzzy strongly UP-ideal of A. Indeed,

$$\varphi^-(3) = -0.2 > -0.7 = \max\{\varphi^-((3 \cdot 4) \cdot (3 \cdot 3)), \varphi^-(4)\}$$

and

$$\varphi^+(3) = 0.1 < 0.5 = \min\{\varphi^+((3 \cdot 4) \cdot (3 \cdot 3)), \varphi^+(4)\}.$$

Theorem 3.13. *Every bipolar fuzzy UP-ideal of A is a bipolar fuzzy UP-filter.*

Proof. Let φ be a bipolar fuzzy UP-ideal of A. Then for all $x, y \in A, \varphi^-(0) \leq \varphi^-(x)$ and

$$\begin{aligned} \varphi^-(y) &= \varphi^-(0 \cdot y) && \text{((UP-2))} \\ &\leq \max\{\varphi^-(0 \cdot (x \cdot y)), \varphi^-(x)\} \\ &= \max\{\varphi^-(x \cdot y), \varphi^-(x)\}, && \text{((UP-2))} \end{aligned}$$

and $\varphi^+(0) \geq \varphi^+(x)$ and

$$\begin{aligned}\varphi^+(y) &= \varphi^+(0 \cdot y) && \text{((UP-2))}\\ &\geq \min\{\varphi^+(0 \cdot (x \cdot y)), \varphi^+(x)\}\\ &= \min\{\varphi^+(x \cdot y), \varphi^+(x)\}. && \text{((UP-2))}\end{aligned}$$

Hence, φ is a bipolar fuzzy UP-filter of A. □

Example 3.14. Consider a UP-algebra $A = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	3
3	0	0	0	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3
φ^-	-0.7	-0.3	-0.1	-0.1
φ^+	0.8	0.5	0.2	0.2

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A but it is not a bipolar fuzzy UP-ideal of A. Indeed,

$$\varphi^-(2 \cdot 3) = \varphi^-(3) = -0.1 > -0.3 = \max\{\varphi^-(2 \cdot (1 \cdot 3)), \varphi^-(1)\}$$

and

$$\varphi^+(2 \cdot 3) = \varphi^+(3) = 0.2 < 0.5 = \min\{\varphi^+(2 \cdot (1 \cdot 3)), \varphi^+(1)\}.$$

Theorem 3.15. *Every bipolar fuzzy UP-filter of A is a bipolar fuzzy UP-subalgebra.*

Proof. Let φ is a bipolar fuzzy UP-filter of A. Then for all $x, y \in A, \varphi^-(0) \leq \varphi^-(x)$ and

$$\begin{aligned}\varphi^-(x \cdot y) &\leq \max\{\varphi^-(y \cdot (x \cdot y)), \varphi^-(y)\}\\ &= \max\{\varphi^-(0), \varphi^-(y)\} && \text{(Proposition 2.2 (5))}\\ &\leq \max\{\varphi^-(x), \varphi^-(y)\},\end{aligned}$$

and $\varphi^+(0) \geq \varphi^+(x)$ and

$$\begin{aligned}\varphi^+(x \cdot y) &\geq \min\{\varphi^+(y \cdot (x \cdot y)), \varphi^+(y)\}\\ &= \min\{\varphi^+(0), \varphi^+(y)\} && \text{(Proposition 2.2 (5))}\\ &\geq \min\{\varphi^+(x), \varphi^+(y)\}.\end{aligned}$$

Hence, φ is a bipolar fuzzy UP-subalgebra of A. □

Example 3.16. Consider a UP-algebra $A = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	0	0	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3
φ^+	0.8	0.4	0.2	0.1
φ^-	−0.9	−0.5	−0.3	−0.2

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A but it is not a bipolar fuzzy UP-filter of A. Indeed,

$$\varphi^-(2) = -0.3 > -0.5 = \max\{\varphi^-(1 \cdot 2), \varphi^-(1)\}$$

and

$$\varphi^+(2) = 0.2 < 0.4 = \min\{\varphi^+(1 \cdot 2), \varphi^+(1)\}.$$

4 Level Cuts of a Bipolar Fuzzy Set

In this section, we discuss the relation between bipolar fuzzy UP-subalgebras (resp., bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals and bipolar fuzzy strongly UP-ideals) and its level cuts.

Definition 4.1. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. For $(t^-, t^+) \in [-1, 0] \times [0, 1]$, the sets

$$N_L(\varphi; t^-) = \{x \in A \mid \varphi^-(x) \leq t^-\}$$

and

$$P_U(\varphi; t^+) = \{x \in A \mid \varphi^+(x) \geq t^+\}$$

are called the *negative lower* t^-*-cut* and the *positive upper* t^+*-cut* of $\varphi = (A; \varphi^-, \varphi^+)$, respectively. The set

$$C(\varphi; (t^-, t^+)) = N_L(\varphi; t^-) \cap P_U(\varphi; t^+)$$

is called the (t^-, t^+)*-cut* of $\varphi = (A; \varphi^-, \varphi^+)$. For any $k \in [0, 1]$, we denote the set

$$C(\varphi; k) = C(\varphi; (-k, k)) = N_L(\varphi; -k) \cap P_U(\varphi; k)$$

is called the k*-cut* of $\varphi = (A; \varphi^-, \varphi^+)$.

Theorem 4.2. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A if and only if the following statements are valid:*

(1) for all $t^- \in [-1, 0], N_L(\varphi; t^-)$ is a UP-subalgebra of A if $N_L(\varphi; t^-)$ is nonempty, and

(2) for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a UP-subalgebra of A if $P_U(\varphi; t^+)$ is nonempty.

Proof. Assume that φ is a bipolar fuzzy UP-subalgebra of A. Let $t^- \in [-1, 0]$ be such that $N_L(\varphi; t^-) \neq \emptyset$ and let $x, y \in N_L(\varphi; t^-)$. Then $\varphi^-(x) \leq t^-$ and $\varphi^-(y) \leq t^-$. Since φ is a bipolar fuzzy UP-subalgebra of A, we have $\varphi^-(x \cdot y) \leq \max\{\varphi^-(x), \varphi^-(y)\} \leq t^-$. Thus $x \cdot y \in N_L(\varphi; t^-)$. Hence, $N_L(\varphi; t^-)$ is a UP-subalgebra of A. Next, let $t^+ \in [0, 1]$ be such that $P_U(\varphi; t^+) \neq \emptyset$ and let $x, y \in P_U(\varphi; t^+)$. Then $\varphi^+(x) \geq t^+$ and $\varphi^+(y) \geq t^+$. Since φ is a bipolar fuzzy UP-subalgebra of A, we have $\varphi^+(x \cdot y) \geq \min\{\varphi^+(x), \varphi^+(y)\} \geq t^+$. Thus $x \cdot y \in P_U(\varphi; t^+)$. Hence, $P_U(\varphi; t^+)$ is a UP-subalgebra of A.

Conversely, assume that for all $t^- \in [0, 1], N_L(\varphi; t^-)$ is a UP-subalgebra of A if $N_L(\varphi; t^-)$ is nonempty and for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a UP-subalgebra of A if $P_U(\varphi; t^+)$ is nonempty. Let $x, y \in A$. Then $\varphi^-(x), \varphi^-(y) \in [-1, 0]$. Choose $t^- = \max\{\varphi^-(x), \varphi^-(y)\}$. Then $\varphi^-(x) \leq t^-$ and $\varphi^-(y) \leq t^-$, that is, $x, y \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a UP-subalgebra of A. So $x \cdot y \in N_L(\varphi; t^-)$. Hence $\varphi^-(x \cdot y) \leq t^- = \max\{\varphi^-(x), \varphi^-(y)\}$. Next, let $x, y \in A$. Then $\varphi^+(x), \varphi^+(y) \in [0, 1]$. Choose $t^+ = \min\{\varphi^+(x), \varphi^+(y)\}$. Then $\varphi^+(x) \geq t^+$ and $\varphi^+(y) \geq t^+$, that is, $x, y \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a UP-subalgebra of A. So $x \cdot y \in P_U(\varphi; t^+)$. Hence, $\varphi^+(x \cdot y) \geq t^+ = \min\{\varphi^+(x), \varphi^+(y)\}$. Therefore, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A. □

Corollary 4.3. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A, then for all $k \in [0, 1], C(\varphi; k)$ is a UP-subalgebra of A while $C(\varphi; k)$ is nonempty.*

Proof. It is straightforward by Theorem 2.7 and 4.2. □

Theorem 4.4. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A if and only if the following statements are valid:*

(1) for all $t^- \in [-1, 0], N_L(\varphi; t^-)$ is a UP-filter of A if $N_L(\varphi; t^-)$ is nonempty, and

(2) for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a UP-filter of A if $P_U(\varphi; t^+)$ is nonempty.

Proof. Assume that φ is a bipolar fuzzy UP-filter of A. Let $t^- \in [-1, 0]$ be such that $N_L(\varphi; t^-) \neq \emptyset$ and let $a \in N_L(\varphi; t^-)$. Then $\varphi^-(a) \leq t^-$. Since φ is a bipolar fuzzy UP-filter of A, we have $\varphi^-(0) \leq \varphi^-(a) \leq t^-$. Thus $0 \in N_L(\varphi; t^-)$. Next, let $x, y \in A$ be such that $x \cdot y \in N_L(\varphi; t^-)$ and $x \in N_L(\varphi; t^-)$. Then $\varphi^-(x \cdot y) \leq t^-$ and $\varphi^-(x) \leq t^-$. Since φ is a bipolar fuzzy UP-filter of A, we have

$$\varphi^-(y) \leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\} \leq t^-.$$

So $y \in N_L(\varphi; t^-)$. Hence, $N_L(\varphi; t^-)$ is a UP-filter of A. Let $t^+ \in [0, 1]$ be such that $P_U(\varphi; t^+) \neq \emptyset$ and let $a \in P_U(\varphi; t^+)$. Then $\varphi^+(a) \geq t^+$. Since φ is a bipolar fuzzy UP-filter of A, we have $\varphi^+(0) \geq \varphi^+(a) \geq t^+$. Thus $0 \in P_U(\varphi; t^+)$. Next, let $x, y \in A$ be such that $x \cdot y \in P_U(\varphi; t^+)$ and $x \in P_U(\varphi; t^+)$. Then $\varphi^+(x \cdot y) \geq t^+$ and $\varphi^+(x) \geq t^+$. Since φ is a bipolar fuzzy UP-filter of A, we have

$$\varphi^+(y) \geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\} \geq t^+.$$

So $y \in P_U(\varphi; t^+)$. Hence, $P_U(\varphi; t^+)$ is a UP-filter of A.

Conversely, assume that for all $t^- \in [0, 1], N_L(\varphi; t^-)$ is a UP-filter of A if $N_L(\varphi; t^-)$ is nonempty and for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a UP-filter of A if $P_U(\varphi; t^+)$ is nonempty. Let $x \in A$. Then $\varphi^-(x) \in [-1, 0]$. Choose $t^- = \varphi^-(x)$. Then $\varphi^-(x) \leq t^-$, that is, $x \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a UP-filter of A. So $0 \in N_L(\varphi; t^-)$. Hence, $\varphi^-(0) \leq t^- = \varphi^-(x)$. Next, let $x, y \in A$. Then $\varphi^-(x \cdot y), \varphi^-(x) \in [-1, 0]$. Choose $t^- = \max\{\varphi^-(x \cdot y), \varphi^-(x)\}$. Then $\varphi^-(x \cdot y) \leq t^-$ and $\varphi^-(x) \leq t^-$, that is, $x \cdot y, x \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a UP-filter of A. So $y \in N_L(\varphi; t^-)$, Hence, $\varphi^-(y) \leq t^- = \max\{\varphi^-(x), \varphi^-(x \cdot y)\}$. Let $x \in A$. Then $\varphi^+(x) \in [0, 1]$. Choose $t^+ = \varphi^+(x)$. Then $\varphi^+(x) \geq t^+$, that is, $x \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a UP-filter of A. So $0 \in P_U(\varphi; t^+)$. Hence, $\varphi^+(0) \geq t^+ = \varphi^+(x)$. Next, let $x, y \in A$. Then $\varphi^+(x \cdot y), \varphi^+(x) \in [0, 1]$. Choose $t^+ = \min\{\varphi^+(x \cdot y), \varphi^+(x)\}$. Then $\varphi^+(x \cdot y) \geq t^+$ and $\varphi^+(x) \geq t^+$, that is, $x \cdot y, x \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a UP-filter of A. So $y \in P_U(\varphi; t^+)$. Hence, $\varphi^+(y) \geq t^+ = \min\{\varphi^+(x), \varphi^+(x \cdot y)\}$. Therefore, φ is a bipolar fuzzy UP-filter of A. □

Corollary 4.5. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A,then for all $k \in [0, 1], C(\varphi; k)$ is a UP-filter of A while $C(\varphi; k)$ is nonempty.*

Proof. It is straightforward by Theorem 2.8 and 4.4. □

Theorem 4.6. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A if and only if the following statements are valid:*

(1) for all $t^- \in [-1, 0], N_L(\varphi; t^-)$ is a UP-ideal of A if $N_L(\varphi; t^-)$ is nonempty, and

(2) for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a UP-ideal of A if $P_U(\varphi; t^+)$ is nonempty.

Proof. Assume that φ is a bipolar fuzzy UP-ideal of A. Let $t^- \in [-1, 0]$ be such that $N_L(\varphi; t^-) \neq \emptyset$ and let $a \in N_L(\varphi; t^-)$. Then $\varphi^-(a) \leq t^-$. Since φ is a bipolar fuzzy UP-ideal of A, we have $\varphi^-(0) \leq \varphi^-(a) \leq t^-$. Thus $0 \in N_L(\varphi; t^-)$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in N_L(\varphi; t^-)$ and $y \in N_L(\varphi; t^-)$. Then $\varphi^-(x \cdot (y \cdot z)) \leq t^-$ and $\varphi^-(y) \leq t^-$. Since φ is a bipolar fuzzy UP-ideal of A, we have

$$\varphi^-(x \cdot z) \leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\} \leq t^-.$$

So $x \cdot z \in N_L(\varphi; t^-)$. Hence, $N_L(\varphi; t^-)$ is a UP-ideal of A. Let $t^+ \in [0, 1]$ be such that $P_U(\varphi; t^+) \neq \emptyset$ and let $a \in P_U(\varphi; t^+)$. Then $\varphi^+(a) \geq t^+$. Since φ is a bipolar fuzzy UP-ideal of A, we have $\varphi^+(0) \geq \varphi^+(a) \geq t^+$. Thus $0 \in P_U(\varphi; t^+)$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in P_U(\varphi; t^+)$ and $y \in P_U(\varphi; t^+)$. Then $\varphi^+(x \cdot (y \cdot z)) \geq t^+$ and $\varphi^+(y) \geq t^+$. Since φ is a bipolar fuzzy UP-ideal of A, we have

$$\varphi^+(x \cdot z) \geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\} \geq t^+.$$

So $x \cdot z \in P_U(\varphi; t^+)$. Hence, $P_U(\varphi; t^+)$ is a UP-ideal of A.

Conversely, assume that for all $t^- \in [0, 1], N_L(\varphi; t^-)$ is a UP-ideal of A if $N_L(\varphi; t^-)$ is nonempty and for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a UP-ideal of A if $P_U(\varphi; t^+)$ is nonempty. Let $x \in A$. Then $\varphi^-(x) \in [-1, 0]$. Choose $t^- = \varphi^-(x)$. Then $\varphi^-(x) \leq t^-$, that is, $x \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a UP-ideal of A. So $0 \in N_L(\varphi; t^-)$. Hence, $\varphi^-(0) \leq t^- = \varphi^-(x)$. Next, let $x, y, z \in A$. Then $\varphi^-(x \cdot (y \cdot z)), \varphi^-(y) \in [-1, 0]$. Choose $t^- = \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\}$. Then $\varphi^-(x \cdot (y \cdot z)) \leq t^-$ and $\varphi^-(y) \leq t^-$, that is, $x \cdot (y \cdot z), y \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a UP-ideal of A. So $x \cdot z \in N_L(\varphi; t^-)$. Hence, $\varphi^-(x \cdot z) \leq t^- = \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\}$. Let $x \in A$. Then $\varphi^+(x) \in [0, 1]$. Choose $t^+ = \varphi^+(x)$. Then $\varphi^+(x) \geq t^+$, that is, $x \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a UP-ideal of A. So $0 \in P_U(\varphi; t^+)$. Hence, $\varphi^+(0) \geq t^+ = \varphi^+(x)$. Next, let $x, y, z \in A$. Then $\varphi^+(x \cdot (y \cdot z)), \varphi^+(y) \in [0, 1]$. Choose $t^+ = \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}$. Then $\varphi^+(x \cdot (y \cdot z)) \geq t^+$ and $\varphi^+(y) \geq t^+$, that is, $x \cdot (y \cdot z), y \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a UP-ideal of A. So $x \cdot z \in P_U(\varphi; t^+)$. Hence, $\varphi^+(x \cdot z) \geq t^+ = \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}$. Therefore, φ is a bipolar fuzzy UP-ideal of A. □

Corollary 4.7. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A, then for all $k \in [0, 1], C(\varphi; k)$ is a UP-ideal of A while $C(\varphi; k)$ is nonempty.*

Proof. It is straightforward by Theorem 2.7 and 4.6. □

Give an example of conflict that the converse of Corollary 4.5, , and 4.7 is not true.

Example 4.8. Consider a UP-algebra $A = \{0, 1, 2, 3\}$ with the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

Define a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A as follows:

A	0	1	2	3
φ^-	-0.6	-0.6	-0.3	-0.6
φ^+	0.6	0.3	0.6	0.3

Then for all $k \in [0,1], C(\varphi; k)$ is a UP-subalgebra (resp., UP-filter, UP-ideal) of A while $C(\varphi; k)$ is nonempty. Indeed,

(1) $C(\varphi; k) = A$ if $k \in [0, 0.3]$,

(2) $C(\varphi; k) = \{0\}$ if $k \in (0.3, 0.6]$, and

(3) $C(\varphi; k) = \emptyset$ if $k \in (0.6, 1]$.

But φ is not a bipolar fuzzy UP-subalgebra of A. Indeed,

$$\varphi^-(1 \cdot 3) = -0.3 > -0.6 = \max\{-0.6, -0.6\} = \max\{\varphi^-(1), \varphi^-(3)\}.$$

By Theorem 3.15 and Theorem 3.13, we have φ is not a bipolar fuzzy UP-filter and a bipolar fuzzy UP-ideal of A.

Theorem 4.9. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A if and only if the following statements are valid:*

(1) for all $t^- \in [-1, 0], N_L(\varphi; t^-)$ is a strongly UP-ideal of A if $N_L(\varphi; t^-)$ is nonempty, and

(2) for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a strongly UP-ideal of A if $P_U(\varphi; t^+)$ is nonempty.

Proof. Assume that φ is a bipolar fuzzy strongly UP-ideal of A. Let $t^- \in [-1, 0]$ be such that $N_L(\varphi; t^-) \neq \emptyset$ and let $a \in N_L(\varphi; t^-)$. Then $\varphi^-(a) \leq t^-$. Since φ is a bipolar fuzzy strongly UP-ideal of A, we have $\varphi^-(0) \leq \varphi^-(a) \leq t^-$. Thus $0 \in N_L(\varphi; t^-)$. Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in N_L(\varphi; t^-)$ and $y \in N_L(\varphi; t^-)$. Then $\varphi^-((z \cdot y) \cdot (z \cdot x)) \leq t^-$ and $\varphi^-(y) \leq t^-$. Since φ is a bipolar fuzzy strongly UP-ideal of A, we have

$$\varphi^-(x) \leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\} \leq t^-.$$

So $x \in N_L(\varphi; t^-)$. Hence, $N_L(\varphi; t^-)$ is a strongly UP-ideal of A. Let $t^+ \in [0, 1]$ be such that $P_U(\varphi; t^+) \neq \emptyset$ and let $a \in P_U(\varphi; t^+)$. Then $\varphi^+(a) \geq t^+$. Since φ is a bipolar fuzzy strongly UP-ideal of A, we have $\varphi^+(0) \geq \varphi^+(a) \geq t^+$. Thus $0 \in P_U(\varphi; t^+)$. Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in P_U(\varphi; t^+)$ and $y \in P_U(\varphi; t^+)$. Then $\varphi^+((z \cdot y) \cdot (z \cdot x)) \geq t^+$ and $\varphi^+(y) \geq t^+$. Since φ is a bipolar fuzzy strongly UP-ideal of A, we have

$$\varphi^+(x) \geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\} \geq t^+.$$

So $x \in P_U(\varphi; t^+)$. Hence, $P_U(\varphi; t^+)$ is a strongly UP-ideal of A.

Conversely, assume that for all $t^- \in [0, 1], N_L(\varphi; t^-)$ is a strongly UP-ideal of A if $N_L(\varphi; t^-)$ is nonempty and for all $t^+ \in [0, 1], P_U(\varphi; t^+)$ is a strongly UP-ideal of A if $P_U(\varphi; t^+)$ is nonempty. Let $x \in A$. Then $\varphi^-(x) \in [-1, 0]$. Choose $t^- = \varphi^-(x)$. Then $\varphi^-(x) \leq t^-$, that is, $x \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a strongly UP-ideal of A. So $0 \in N_L(\varphi; t^-)$. Hence, $\varphi^-(0) \leq t^- = \varphi^-(x)$. Next, let $x, y, z \in A$. Then $\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y) \in [-1, 0]$. Choose $t^- = \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$. Then $\varphi^-((z \cdot y) \cdot (z \cdot x)) \leq t^-$ and $\varphi^-(y) \leq t^-$, that is, $(z \cdot y) \cdot (z \cdot x), y \in N_L(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_L(\varphi; t^-)$ is a strongly UP-ideal of A. So $x \in N_L(\varphi; t^-)$. Hence, $\varphi^-(x) \leq t^- = \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$. Let $x \in A$. Then $\varphi^+(x) \in [0, 1]$. Choose $t^+ =$

$\varphi^+(x)$. Then $\varphi^+(x) \geq t^+$, that is, $x \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a strongly UP-ideal of A. So $0 \in P_U(\varphi; t^+)$. Hence, $\varphi^+(0) \geq t^+ = \varphi^+(x)$. Next, let $x, y, z \in A$. Then $\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y) \in [0, 1]$. Choose $t^+ = \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}$. Then $\varphi^+((z \cdot y) \cdot (z \cdot x)) \geq t^+$ and $\varphi^+(y) \geq t^+$, that is, $(z \cdot y) \cdot (z \cdot x), y \in P_U(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_U(\varphi; t^+)$ is a strongly UP-ideal of A. So $x \in P_U(\varphi; t^+)$. Hence, $\varphi^+(x) \geq t^+ = \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}$. Therefore, φ is a bipolar fuzzy strongly UP-ideal of A. □

Corollary 4.10. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A if and only if for all $k \in [0, 1], C(\varphi; k)$ is a strongly UP-ideal of A while $C(\varphi; k)$ is nonempty.*

Proof. It is straightforward by Theorem 2.9, 4.9, and 3.10, and A is the only one strongly UP-ideal of itself. □

Theorem 4.11. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-filter of A satisfies the following assertion:*

$$x \cdot (y \cdot z) = y \cdot (x \cdot z) \text{ for all } x, y, z \in A. \tag{4.1}$$

Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A.

Proof. For all $x, y, z \in A$,

$$\varphi^-(0) \leq \varphi^-(x)$$

and

$$\begin{aligned} \varphi^-(x \cdot z) &\leq \max\{\varphi^-(y \cdot (x \cdot z)), \varphi^-(y)\} \\ &= \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\}, && ((4.1)) \end{aligned}$$

and

$$\varphi^+(0) \geq \varphi^+(x)$$

and

$$\begin{aligned} \varphi^+(x \cdot z) &\geq \min\{\varphi^+(y \cdot (x \cdot z)), \varphi^+(y)\} \\ &= \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}. && ((4.1)) \end{aligned}$$

Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A. □

Definition 4.12. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. We define a subset $\varphi^{-1}(0, 0)$ of A by

$$\varphi^{-1}(0, 0) = \{x \in A \mid \varphi^-(x) = \varphi^-(0) \text{ and } \varphi^+(x) = \varphi^+(0)\}.$$

Theorem 4.13. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-subalgebra of A. Then $\varphi^{-1}(0, 0)$ is a UP-subalgebra of A.*

Proof. Clearly, $0 \in \varphi^{-1}(0, 0)$. Let $x, y \in \varphi^{-1}(0, 0)$. Then $\varphi^-(x) = \varphi^-(0), \varphi^+(x) = \varphi^+(0), \varphi^-(y) = \varphi^-(0)$, and $\varphi^+(y) = \varphi^+(0)$. Thus

$$\begin{aligned} \varphi^-(0) &\leq \varphi^-(x \cdot y) \\ &\leq \max\{\varphi^-(x), \varphi^-(y)\} \\ &= \max\{\varphi^-(0), \varphi^-(0)\} \\ &= \varphi^-(0) \end{aligned}$$

and

$$\begin{aligned}\varphi^+(0) &\geq \varphi^+(x \cdot y) \\ &\geq \min\{\varphi^+(x), \varphi^+(y)\} \\ &= \min\{\varphi^+(0), \varphi^+(0)\} \\ &= \varphi^+(0).\end{aligned}$$

So $\varphi^-(x \cdot y) = \varphi^-(0)$ and $\varphi^+(x \cdot y) = \varphi^+(0)$, that is, $x \cdot y \in \varphi^{-1}(0, 0)$. Therefore, $\varphi^{-1}(0, 0)$ is a UP-subalgebra of A. □

Theorem 4.14. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-filter of A. Then $\varphi^{-1}(0, 0)$ is a UP-filter of A.*

Proof. Clearly, $0 \in \varphi^{-1}(0, 0)$. Let $x, y \in A$ be such that $x \cdot y \in \varphi^{-1}(0, 0)$ and $x \in \varphi^{-1}(0, 0)$. Then $\varphi^-(x) = \varphi^-(0), \varphi^+(x) = \varphi^+(0), \varphi^-(x \cdot y) = \varphi^-(0)$, and $\varphi^+(x \cdot y) = \varphi^+(0)$. Thus

$$\begin{aligned}\varphi^-(0) &\leq \varphi^-(y) \\ &\leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\} \\ &= \max\{\varphi^-(0), \varphi^-(0)\} \\ &= \varphi^-(0)\end{aligned}$$

and

$$\begin{aligned}\varphi^+(0) &\geq \varphi^+(y) \\ &\geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\} \\ &= \min\{\varphi^+(0), \varphi^+(0)\} \\ &= \varphi^+(0).\end{aligned}$$

So $\varphi^-(y) = \varphi^-(0)$ and $\varphi^+(y) = \varphi^+(0)$, that is, $y \in \varphi^{-1}(0, 0)$. Therefore, $\varphi^{-1}(0, 0)$ is a UP-filter of A. □

Theorem 4.15. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-ideal of A. Then $\varphi^{-1}(0, 0)$ is a UP-ideal of A.*

Proof. Clearly, $0 \in \varphi^{-1}(0, 0)$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in \varphi^{-1}(0, 0)$ and $y \in \varphi^{-1}(0, 0)$. Then $\varphi^-(x \cdot (y \cdot z)) = \varphi^-(0), \varphi^+(x \cdot (y \cdot z)) = \varphi^+(0), \varphi^-(y) = \varphi^-(0)$, and $\varphi^+(y) = \varphi^+(0)$. Thus

$$\begin{aligned}\varphi^-(0) &\leq \varphi^-(x \cdot z) \\ &\leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\} \\ &= \max\{\varphi^-(0), \varphi^-(0)\} \\ &= \varphi^-(0)\end{aligned}$$

and

$$\begin{aligned}\varphi^+(0) &\geq \varphi^+(x \cdot z) \\ &\geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\} \\ &= \min\{\varphi^+(0), \varphi^+(0)\} \\ &= \varphi^+(0).\end{aligned}$$

So $\varphi^-(x \cdot z) = \varphi^-(0)$ and $\varphi^+(x \cdot z) = \varphi^+(0)$, that is, $x \cdot z \in \varphi^{-1}(0, 0)$. Therefore, $\varphi^{-1}(0, 0)$ is a UP-ideal of A. □

Give an example of conflict that the converse of Theorem 4.13, 4.14, and 4.15 is not true.

Example 4.16. From Example 4.8, we have $\varphi^{-1}(0,0) = \{0\}$ is a UP-subalgebra (resp., UP-filter, UP-ideal) of A but φ is not a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-ideal, bipolar fuzzy UP-filter) of A.

Theorem 4.17. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A if and only if $\varphi^{-1}(0,0)$ is a strongly UP-ideal of A.*

Proof. It is straightforward by Theorem 3.10, and A is the only one strongly UP-ideal of itself. □

Acknowledgment

The authors wish to express their sincere thanks to the referees for the valuable suggestions which lead to an improvement of this paper.

References

[1] T. Guntasow, S. Sajak, A. Jomkham, and A. Iampan, *Fuzzy translations of a fuzzy set in UP-algebras*, J. Indones. Math. Soc. **23** (2017), no. 2, 1–19.

[2] Q. P. Hu and X. Li, *On BCH-algebras*, Math. Semin. Notes, Kobe Univ. **11** (1983), 313–320.

[3] A. Iampan, *A new branch of the logical algebra: UP-algebras*, J. Algebra Relat. Top. **5** (2017), no. 1, 35–54.

[4] Y. Imai and K. Iséki, *On axiom system of propositional calculi, XIV*, Proc. Japan Acad. **42** (1966), no. 1, 19–22.

[5] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Acad. **42** (1966), no. 1, 26–29.

[6] Y. B. Jun, K. J. Lee, and E. H. Roh, *Ideals and filters in CI-algebras based on bipolar-valued fuzzy sets*, Ann. Fuzzy Math. Inform. **4** (2012), no. 1, 109–121.

[7] Y. B. Jun and C. H. Park, *Filters of BCH-algebras based on bipolar-valued fuzzy sets*, Int. Math. Forum **4** (2009), no. 13, 631–643.

[8] Y. B. Jun and S. Z. Song, *Subalgebras and closed ideals of BCH-algebras based on bipolar-valued fuzzy sets*, Sci. Math. Jpn **68** (2008), no. 2, 287–297.

[9] S. Keawrahun and U. Leerawat, *On isomorphisms of SU-algebras*, Sci. Magna **7** (2011), no. 2, 39–44.

[10] K. J. Lee and Y. B. Jun, *Bipolar fuzzy a-ideals of BCI-algebras*, Commun. Korean Math. Soc. **26** (2011), no. 4, 531–542.

[11] K. M. Lee, *Bipolar-valued fuzzy sets and their operations*, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand (2000), 307–312.

[12] G. Muhiuddin, *Bipolar fuzzy KU-subalgebras/ideals of KU-algebras*, Ann. Fuzzy Math. Inform. **8** (2014), no. 3, 409–418.

[13] C. Prabpayak and U. Leerawat, *On ideals and congruences in KU-algebras*, Sci. Magna **5** (2009), no. 1, 54–57.

[14] S. Sabarinathan, D. C. Kumar, and P. Muralikrishna, *Bipolar valued fuzzy α-ideal of BF-algebra*, Circuits Syst. **7** (2016), 3054–3062.

[15] ———, *Bipolar valued fuzzy ideals of BF-algebras*, Int. J. Pure Appl. Math. **109** (2016), no. 4, 837–846.

[16] S. Sabarinathan, P. Muralikrishna, and D. C. Kumar, *Bipolar valued fuzzy H-ideals of BF-algebras*, Int. J. Pure Appl. Math. **112** (2017), no. 5, 87–92.

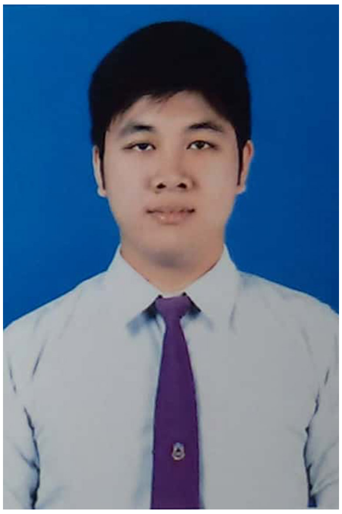
[17] T. Senapati, *Bipolar fuzzy structure of BG-subalgebras*, J. Fuzzy Math. **23** (2015), no. 1, 209–220.

[18] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, and A. Iampan, *Fuzzy sets in UP-algebras*, Ann. Fuzzy Math. Inform. **12** (2016), no. 6, 739–756.

[19] L. A. Zadeh, *Fuzzy sets*, Inf. Cont. **8** (1965), 338–353.

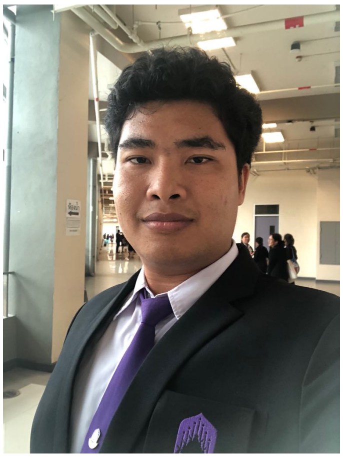
BIOGRAPHY

BIOGRAPHY



Name Surname	Korawut Kawila
Date of Birth	30 May 1996
Education Background	Junior High School from Nawamindarajudid Machim School in 2010, Muang, Nakhonsawan, Thailand Senior High School from Nawamindarajudid Machim School in 2013, Muang, Nakhonsawan, Thailand
Address	House No. 100/21 Village No. 2 Phayuha sub-district, Phayuha Khiri district, Nakhonsawan province 60130
Telephone Number	0873063645
E-mail	korawutkawila@gmail.com

BIOGRAPHY



Name Surname	Chaiphon Udomsetchai
Date of Birth	25 June 1996
Education Background	Junior High School from Muangchaliang School in 2010, Si Satchanalai, Sukhothai, Thailand Senior High School from Muangchaliang School in 2013, Si Satchanalai, Sukhothai, Thailand
Address	House No. 411 Village No. 21 Mae Sin sub-district, Si Satchanalai district, Sukhothai province 64130
Telephone Number	0802082538
E-mail	lovesumen3@gmail.com

Bipolar Fuzzy UP-Algebras 2018